IMO National Selection Test
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IMONST 2 2021
JUNIOR CATEGORY

Solution

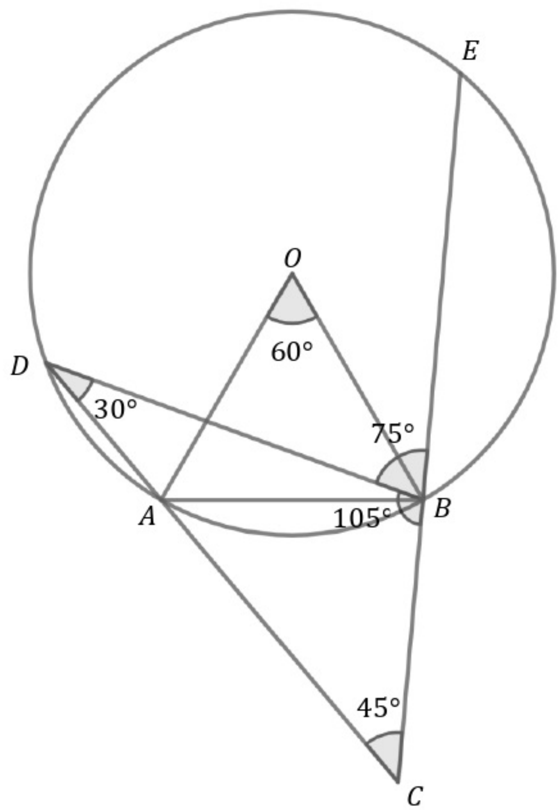
Problem 1

Given a circle with center O. Points A and B lie on the circle such that triangle OBA is equilateral. Let C be a point outside the circle with $\angle ACB = 45^\circ$. Line CA intersects the circle at point D, and line CB intersects the circle at point E.

Find $\angle DBE$.

Proof. Sketch the circle with center O and the triangle OAB such that the side AB is below O. Since OAB is an equilateral triangle, we know $\angle AOB = 60^\circ$. Any angle subtended by the arc AB at the circumference is half of that at the center, which is 30°. Since $\angle ACB = 45^\circ$ which is bigger than 30°, point C must be somewhere below the line AB. We continue to sketch the remaining points and lines as shown below.

Angle $\angle ADB$ is subtended by the arc AB at the circumference, so $\angle ADB = 30^\circ$. Since the sum of angles in any triangle including BCD is 180°, we have $\angle CBD = 105^\circ$. Since CBE is a straight line, we have $\angle CBD + \angle DBE = 180^\circ$, and thus $\angle DBE = 75^\circ$. □



Problem 2

Can we find positive integers a and b such that both $a^2 + b$ and $b^2 + a$ are perfect squares?

Proof. Suppose that such a and b exist. The perfect square $a^2 + b$ is larger than the perfect square a^2. The next perfect square after a^2 is $(a+1)^2$. So, $a^2 + b$ is either $(a+1)^2$ itself or more. This implies $a^2 + b \geq (a+1)^2$. The previous inequality is simplified to be $b \geq 2a + 1$.

We argue similarly to obtain $b^2 + a \geq (b+1)^2$ and thus $a \geq 2b + 1$. Adding both inequalities, we have $a + b \geq 2a + 2b + 2$, or equivalently, $a + b \leq -2$. This contradicts the fact that a and b are positive and so is their sum $a + b$.

So, such positive integers a and b cannot exist. □

Problem 3

Given a sequence of positive integers

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

such that $a_2 > a_1$ and $a_{n+2} = 3a_{n+1} - 2a_n$ for all $n \geq 1$.

Prove that $a_{2021} > 2^{2019}$.

Proof. Equation $a_{n+2} = 3a_{n+1} - 2a_n$ can be written as $a_{n+2} - a_{n+1} = 2\left(a_{n+1} - a_n\right)$. Using the latter, we can write the terms as follows:

$$\begin{aligned}
n &= 1: & a_3 - a_2 &= 2\left(a_2 - a_1\right), \\
n &= 2: & a_4 - a_3 &= 2\left(a_3 - a_2\right), \\
n &= 3: & a_5 - a_4 &= 2\left(a_4 - a_3\right), \\
&\dots \\
n &= 2019: & a_{2021} - a_{2020} &= 2\left(a_{2020} - a_{2019}\right).
\end{aligned}$$

We can do consecutive substitution as follows:

$$\begin{aligned}
a_{2021} - a_{2020} &= 2\left(a_{2020} - a_{2019}\right) \\
&= 2^2\left(a_{2019} - a_{2018}\right) \\
&= 2^3\left(a_{2018} - a_{2017}\right) \\
&= \dots \\
&= 2^{2019}\left(a_2 - a_1\right).
\end{aligned}$$

Overall, we have

$$a_{2021} = 2^{2019}\left(a_2 - a_1\right) + a_{2020}.$$

Now, a_1 and a_2 are positive integers such that $a_2 > a_1$. This implies that a_2 must either be $a_1 + 1$ or more. In other words, $a_2 \geq a_1 + 1$ or equivalently, $a_2 - a_1 \geq 1$. Since a_{2020} is positive, we have $a_{2020} > 0$. Using the last two inequalities on the equation above, we deduce that $a_{2021} > 2^{2019}$. □

Problem 4

Find all values of n such that there exists a rectangle with integer side lengths, perimeter n, and area $2n$.

Proof. Denote a and b as the length and width of such rectangle. Calculations on perimeter and area of the rectangle give us

$$2a + 2b = n \quad \text{and} \quad ab = 2n$$

which then implies

$$ab - 4a - 4b = 0.$$

We do factorisation by applying the following trick:

$$\begin{aligned}
& ab - 4a - 4b + 16 = 16 \\
\implies & (a-4)(b-4) = 16.
\end{aligned}$$

Note that a and b are integers, and so are $a-4$ and $b-4$. Both expressions are factors of 16, which can be positive or negative factors. The possible pairs of values for $(a-4, b-4)$ are as follows: $(1,16), (2,8), (4,4), (8,2), (16,1), (-1,-16), (-2,-8), (-4,-4), (-8,-2), (-16,-1)$. Solving for each pair, we have the positive integer for (a, b) as follows: $(5,20)$, $(6,12)$, $(8,8)$, $(12,6)$, $(20,5)$. Hence, the possible values of n are 32, 36 and 50. □

Problem 5

Six teams participate in a hockey tournament. Each team plays once against every other team. In each game, a team is awarded 3 points for a win, 1 point for a draw, and 0 point for a loss.

After the tournament, the teams are ranked by total points. No two teams have the same total points. Each team (except the bottom team) has 2 points more than the team ranking one place lower.

Prove that the team that finished fourth has won two games and lost three games.

Proof. Denote T_1, T_2 and up till T_6 as the team ranking first, second and so on respectively. Denote p as the total points of the last team T_6. Based on the given conditions, the sum of points for all teams is $6p + 30$.

A game can end up with a win-lose which contributes a total 3 points to the sum, or a draw which contributes a total 2 points to the sum. Since there are 15 games, the maximum sum is 45 which is achieved when each game ends up with win-lose. So, we have $6p + 30 \leq 45$ which then implies that $p \leq 2$.

We check which values of p is possible.

(a) $p = 0$
Based on the conditions, T_5 has 2 points. Note that T_6 loses in each game, including against T_5. This implies that T_5 has at least 3 points, which is a contradiction.

(b) $p = 1$
The sum of points for all teams is 36. Let a and b be the number of games ending up with win-lose and draw respectively. Based on the number of games and also point contribution, we have $a + b = 15$ and $3a + 2b = 36$. This implies that $a = 6$ and $b = 9$. We focus on the games ending up with win-lose, in which there shall be 6 such games.

Based on points, T_6 has a draw and four losses. T_5 has either a win and four losses, or three draws and two losses. In the first case for T_5, the win must come from the game against T_6. Overall in this case, the following games end up with win-lose: T_6 against each team except T_5, and T_5 against each team except T_6.

In the second case for T_5, a draw must come from the game against T_6. So, T_6 loses against each team other than T_5, and that includes T_4. Since T_4 has 5 points, it must end up with a win, two draws and two loses. Consequently for T_5, another draw must come from the game against T_4. Overall in this case, the following games end up with win-lose: T_6 against each team except T_5, two games in which T_5 loses (that are not against T_6 and T_4) and two games in which T_4 loses (that are not against T_6 and T_5).

In both cases, there are at least 8 games ending up with win-lose in the tournament. This contradicts our earlier calculation.

(c) $p = 2$
Using similar calculation in (ii), there are 12 games and 3 games ending up with win-

lose and draw respectively in the tournament. We focus on the games ending up with draw, in which there shall be 3 such games.

Based on the points, T_6 has exactly two draws, T_5 and T_2 each has at least one draw and T_3 has at least two draws. So, those three games which end up with draw must involve these teams only. Since T_4 has 6 points and does not have any draw, it must have two wins and three losses.

Overall, we deduce that

(i) T_1 has four wins and a loss,

(ii) T_2 has three wins, a draw and two losses,

(iii) T_3 has two wins, two draws and a loss,

(iv) T_4 has two wins and three losses,

(v) T_5 has a win, a draw and three losses, and

(vi) T_6 has two draws and three losses.

We have to check whether this outcome is possible in the tournament. Indeed, one such example is shown below.

Team	Wins against	Draws against	Loses against
T_1	T_2, T_3, T_4, T_6	None	T_5
T_2	T_4, T_5, T_6	T_3	T_1
T_3	$T_4, T_5,$	T_2, T_6	T_1
T_4	T_5, T_6	None	T_1, T_2, T_3
T_5	T_1	T_6	T_2, T_3, T_4
T_6	None	T_3, T_5	T_1, T_2, T_4

We have proved that every possible outcome of the tournament always leads to T_4 having two wins and three losses. □

Problem 6

Let x and y be two rational numbers such that

$$x^5 + y^5 = 2x^2y^2.$$

Prove that $\sqrt{1 - xy}$ is also a rational number.

Note: A rational number is a number that can be expressed as $\frac{a}{b}$ where a and b are integers ($b \neq 0$).

Proof. If either $x = 0$ and $y = 0$, then we have $\sqrt{1 - xy} = 1$ which is clearly rational. To avoid such trivial case, we assume that x and y are non-zero. Observe the following trick:

$$\begin{aligned} x^5 + y^5 = 2x^2y^2 &\implies (x^5 + y^5)^2 = 4x^4y^4 \\ &\implies x^{10} + y^{10} + 2x^5y^5 = 4x^4y^4 \\ &\implies (x^5 - y^5)^2 + 4x^5y^5 = 4x^4y^4 \\ &\implies (x^5 - y^5)^2 = 4x^4y^4(1 - xy) \\ &\implies 1 - xy = \frac{(x^5 - y^5)^2}{4x^4y^4} \\ &\implies \sqrt{1 - xy} = \frac{|x^5 - y^5|}{2x^2y^2}. \end{aligned}$$

Since x and y are rational and the arithmetic operations preserve rational property of numbers, the right-hand side of the equation is rational, and so is $\sqrt{1 - xy}$. $\square$