IMO National Selection Test
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PRIMARY CATEGORY

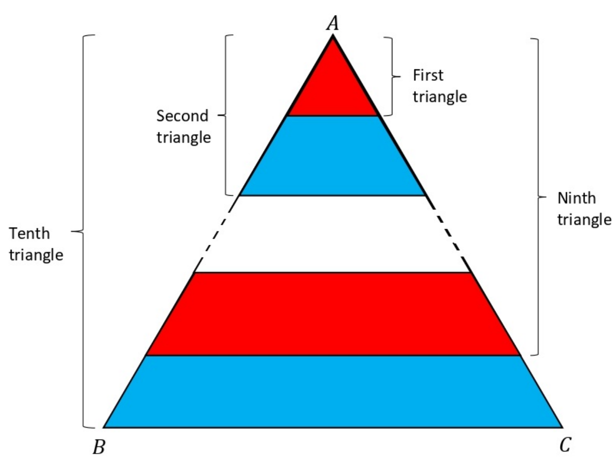
Solution

Problem 1

An equilateral triangle ABC is divided by nine lines parallel to BC into ten bands that are equally wide. We colour the bands alternately red and blue, with the smallest band coloured red. The difference between the total area in red and the total area in blue is 20 cm^2.

What is the area of triangle ABC?

Proof. Consider the following diagram. Denote the area of the first triangle as x. Since each



band has equal wide, the height of the second triangle is twice the height of the first one, and the same is true for its base too. By calculation, the area of the second triangle is $4x$.

Similarly, the height and base of the third triangle are thrice the height and base of the first one, so its area is $9x$. Continuing this way, the areas of the triangles in this order are given as follows: $x, 4x, 9x, 16x, \ldots, 81x, 100x$.

Now, the second band is obtained by removing the first triangle from the second one. So, the area of the second band is $3x$. Similarly, the third band is obtained by removing the second triangle from the third one, so its area is $5x$. Continuing this way, the areas of the bands in this order are given as follows: $x, 3x, 5x, 7x, \ldots, 17x, 19x$.

Overall, the areas of the red bands are $x, 5x, 9x, 13x$ and $17x$, with total area $45x$. The areas of the blue bands are $3x, 7x, 11x, 15x$ and $19x$, with total area $55x$. The difference between the total areas is $10x$. It is given that this difference is 20 cm^2, so $x = 2$ cm^2.

The area of triangle ABC is $100x$, which is 200 cm^2. □

Problem 2

The five numbers a, b, c, d and e satisfy the inequalities

$$a + b < c + d < e + a < b + c < d + e.$$

Among the five numbers, which one is the smallest, and which one is the largest?

Proof. From the inequalities, we pair up expressions containing the same number. We have

(i) $a + b < e + a$ and so $b < e$,

(ii) $a + b < b + c$ and so $a < c$,

(iii) $c + d < b + c$ and so $d < b$,

(iv) $c + d < d + e$ and so $c < e$, and

(v) $e + a < d + e$ and so $a < d$

by cancelling the same number in each inequality. Combining these inequalities, we obtain

$$a < c < e \quad \text{and} \quad a < d < b < e.$$

So, the smallest number is a, and the largest number is e. □

Problem 3

Given a cube. On each edge of the cube, we write a number, either 1 or -1. For each face of the cube, we multiply the four numbers on the edges of this face, and write the product on this face. Finally, we add all the eighteen numbers that we wrote down on the edges and faces of the cube.

What is the smallest possible sum that we can get?

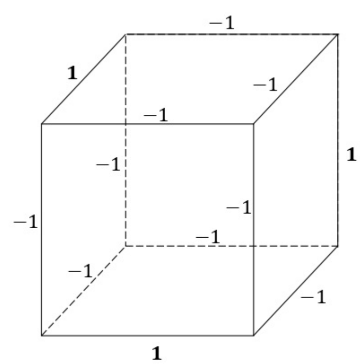
Proof. Note that a face has number -1 if and only if exactly one or three of its edges have number -1. For the smallest possible sum, the idea is to increase the number of -1's and reduce the number of 1's on the cube. We can check each possible configuration as follows:

(i) eighteen -1's and no 1. This means that all edges and faces have number -1. However, if all edges have number -1, then all faces have number 1. This is a contradiction.

(ii) seventeen -1's and one 1. If the number 1 is on an edge, then any face without that edge will have number 1. If it is on a face, then all edges have number -1 and thus, other faces have number 1. Both cases contradict that there is only one 1.

(iii) sixteen -1's and two 1's. If the two 1's are both on edges, then any face without these edges will have number 1. If they are both on faces, then all edges have number -1 and thus, other faces have number 1. If they are on an edge and a face, then any other face without that edge will have number 1. All cases contradict that there are only two 1's.

(iv) fifteen -1's and three 1's. This is possible by setting three 1's on certain edges as shown below. The sum is -12.



Since it is impossible to go lower as explained above, the smallest possible sum is -12. $\square$

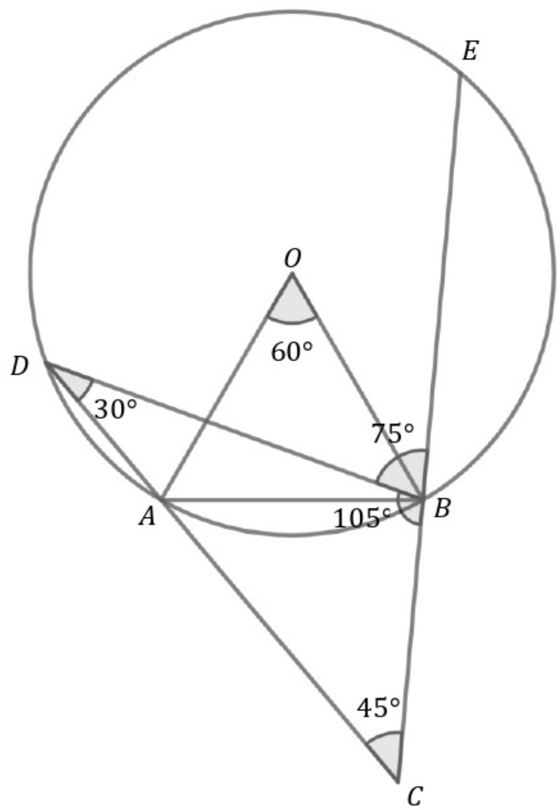
Problem 4

Given a circle with center O. Points A and B lie on the circle such that triangle OBA is equilateral. Let C be a point outside the circle with $\angle ACB = 45°$. Line CA intersects the circle at point D, and line CB intersects the circle at point E.

Find $\angle DBE$.

Proof. Sketch the circle with center O and the triangle OAB such that the side AB is below O. Since OAB is an equilateral triangle, we know $\angle AOB = 60°$. Any angle subtended by the arc AB at the circumference is half of that at the center, which is $30°$. Since $\angle ACB = 45°$ which is bigger than $30°$, point C must be somewhere below the line AB. We continue to sketch the remaining points and lines as shown below.

Angle $\angle ADB$ is subtended by the arc AB at the circumference, so $\angle ADB = 30°$. Since the sum of angles in any triangle including BCD is $180°$, we have $\angle CBD = 105°$. Since CBE is a straight line, we have $\angle CBD + \angle DBE = 180°$, and thus $\angle DBE = 75°$. $\square$



Problem 5

Can we find positive integers a and b such that both $a^2 + b$ and $b^2 + a$ are perfect squares?

Proof. Suppose that such a and b exist. The perfect square $a^2 + b$ is larger than the perfect square a^2. The next perfect square after a^2 is $(a + 1)^2$. So, $a^2 + b$ is either $(a + 1)^2$ itself or more. This implies $a^2 + b \geq (a + 1)^2$. The previous inequality is simplified to be $b \geq 2a + 1$.

We argue similarly to obtain $b^2 + a \geq (b + 1)^2$ and thus $a \geq 2b + 1$. Adding both inequalities, we have $a + b \geq 2a + 2b + 2$, or equivalently, $a + b \leq -2$. This contradicts the fact that a and b are positive and so is their sum $a + b$.

So, such positive integers a and b cannot exist. □

Problem 6

Given a sequence of positive integers

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

such that $a_2 > a_1$ and $a_{n+2} = 3a_{n+1} - 2a_n$ for all $n \geq 1$.

Prove that $a_{2021} > 2^{2019}$.

Proof. Equation $a_{n+2} = 3a_{n+1} - 2a_n$ can be written as $a_{n+2} - a_{n+1} = 2\left(a_{n+1} - a_n\right)$. Using the latter, we can write the terms as follows:

$$\begin{aligned}
&n = 1: && a_3 - a_2 = 2\left(a_2 - a_1\right), \\
&n = 2: && a_4 - a_3 = 2\left(a_3 - a_2\right), \\
&n = 3: && a_5 - a_4 = 2\left(a_4 - a_3\right), \\
&\dots \\
&n = 2019: && a_{2021} - a_{2020} = 2\left(a_{2020} - a_{2019}\right).
\end{aligned}$$

We can do consecutive substitution as follows:

$$\begin{aligned}
a_{2021} - a_{2020} &= 2\left(a_{2020} - a_{2019}\right) \\
&= 2^2\left(a_{2019} - a_{2018}\right) \\
&= 2^3\left(a_{2018} - a_{2017}\right) \\
&= \dots \\
&= 2^{2019}\left(a_2 - a_1\right).
\end{aligned}$$

Overall, we have

$$a_{2021} = 2^{2019}\left(a_2 - a_1\right) + a_{2020}.$$

Now, a_1 and a_2 are positive integers such that $a_2 > a_1$. This implies that a_2 must either be $a_1 + 1$ or more. In other words, $a_2 \geq a_1 + 1$ or equivalently, $a_2 - a_1 \geq 1$. Since a_{2020} is positive, we have $a_{2020} > 0$. Using the last two inequalities on the equation above, we deduce that $a_{2021} > 2^{2019}$. □