IMO National Selection Test
Ujian Pemilihan Kebangsaan IMO

IMONST 2 2021
SENIOR CATEGORY

Solution

Problem 1

Find all values of n such that there exists a rectangle with integer side lengths, perimeter n, and area $2n$.

Proof. Denote a and b as the length and width of such rectangle. Calculations on perimeter and area of the rectangle give us

$$2a + 2b = n \quad \text{and} \quad ab = 2n$$

which then implies

$$ab - 4a - 4b = 0.$$

We do factorisation by applying the following trick:

$$\begin{aligned} &ab - 4a - 4b + 16 = 16 \\ \implies &(a-4)(b-4) = 16. \end{aligned}$$

Note that a and b are integers, and so are $a-4$ and $b-4$. Both expressions are factors of 16, which can be positive or negative factors. The possible pairs of values for $(a-4, b-4)$ are as follows: $(1,16), (2,8), (4,4), (8,2), (16,1), (-1,-16), (-2,-8), (-4,-4), (-8,-2), (-16,-1)$. Solving for each pair, we have the positive integer for (a, b) as follows: $(5, 20)$, $(6, 12)$, $(8, 8)$, $(12, 6)$, $(20, 5)$. Hence, the possible values of n are 32, 36 and 50. □

Problem 2

Six teams participate in a hockey tournament. Each team plays once against every other team. In each game, a team is awarded 3 points for a win, 1 point for a draw, and 0 point for a loss.

After the tournament, the teams are ranked by total points. No two teams have the same total points. Each team (except the bottom team) has 2 points more than the team ranking one place lower.

Prove that the team that finished fourth has won two games and lost three games.

Proof. Denote T_1, T_2 and up till T_6 as the team ranking first, second and so on respectively. Denote p as the total points of the last team T_6. Based on the given conditions, the sum of points for all teams is $6p + 30$.

A game can end up with a win-lose which contributes a total 3 points to the sum, or a draw which contributes a total 2 points to the sum. Since there are 15 games, the maximum sum is 45 which is achieved when each game ends up with win-lose. So, we have $6p + 30 \leq 45$ which then implies that $p \leq 2$.

We check which values of p is possible.

(a) $p = 0$
Based on the conditions, T_5 has 2 points. Note that T_6 loses in each game, including against T_5. This implies that T_5 has at least 3 points, which is a contradiction.

(b) $p = 1$
The sum of points for all teams is 36. Let a and b be the number of games ending up with win-lose and draw respectively. Based on the number of games and also point contribution, we have $a + b = 15$ and $3a + 2b = 36$. This implies that $a = 6$ and $b = 9$. We focus on the games ending up with win-lose, in which there shall be 6 such games.

Based on points, T_6 has a draw and four losses. T_5 has either a win and four losses, or three draws and two losses. In the first case for T_5, the win must come from the game against T_6. Overall in this case, the following games end up with win-lose: T_6 against each team except T_5, and T_5 against each team except T_6.

In the second case for T_5, a draw must come from the game against T_6. So, T_6 loses against each team other than T_5, and that includes T_4. Since T_4 has 5 points, it must end up with a win, two draws and two loses. Consequently for T_5, another draw must come from the game against T_4. Overall in this case, the following games end up with win-lose: T_6 against each team except T_5, two games in which T_5 loses (that are not against T_6 and T_4) and two games in which T_4 loses (that are not against T_6 and T_5).

In both cases, there are at least 8 games ending up with win-lose in the tournament. This contradicts our earlier calculation.

(c) $p = 2$
Using similar calculation in (ii), there are 12 games and 3 games ending up with win-

lose and draw respectively in the tournament. We focus on the games ending up with draw, in which there shall be 3 such games.

Based on the points, T_6 has exactly two draws, T_5 and T_2 each has at least one draw and T_3 has at least two draws. So, those three games which end up with draw must involve these teams only. Since T_4 has 6 points and does not have any draw, it must have two wins and three losses.

Overall, we deduce that

(i) T_1 has four wins and a loss,

(ii) T_2 has three wins, a draw and two losses,

(iii) T_3 has two wins, two draws and a loss,

(iv) T_4 has two wins and three losses,

(v) T_5 has a win, a draw and three losses, and

(vi) T_6 has two draws and three losses.

We have to check whether this outcome is possible in the tournament. Indeed, one such example is shown below.

Team	Wins against	Draws against	Loses against
T_1	T_2, T_3, T_4, T_6	None	T_5
T_2	T_4, T_5, T_6	T_3	T_1
T_3	T_4, T_5	T_2, T_6	T_1
T_4	T_5, T_6	None	T_1, T_2, T_3
T_5	T_1	T_6	T_2, T_3, T_4
T_6	None	T_3, T_5	T_1, T_2, T_4

We have proved that every possible outcome of the tournament always leads to T_4 having two wins and three losses. $\square$

Problem 3

Let x and y be two rational numbers such that

$$x^5 + y^5 = 2x^2y^2.$$

Prove that $\sqrt{1 - xy}$ is also a rational number.

Note: A rational number is a number that can be expressed as $\frac{a}{b}$ where a and b are integers $(b \neq 0)$.

Proof. If either $x = 0$ and $y = 0$, then we have $\sqrt{1-xy} = 1$ which is clearly rational. To avoid such trivial case, we assume that x and y are non-zero. Observe the following trick:

$$\begin{aligned} x^5 + y^5 = 2x^2y^2 &\implies (x^5+y^5)^2 = 4x^4y^4 \\ &\implies x^{10} + y^{10} + 2x^5y^5 = 4x^4y^4 \\ &\implies (x^5-y^5)^2 + 4x^5y^5 = 4x^4y^4 \\ &\implies (x^5-y^5)^2 = 4x^4y^4(1-xy) \\ &\implies 1 - xy = \frac{(x^5-y^5)^2}{4x^4y^4} \\ &\implies \sqrt{1-xy} = \frac{|x^5-y^5|}{2x^2y^2}. \end{aligned}$$

Since x and y are rational and the arithmetic operations preserve rational property of numbers, the right-hand side of the equation is rational, and so is $\sqrt{1-xy}$. □

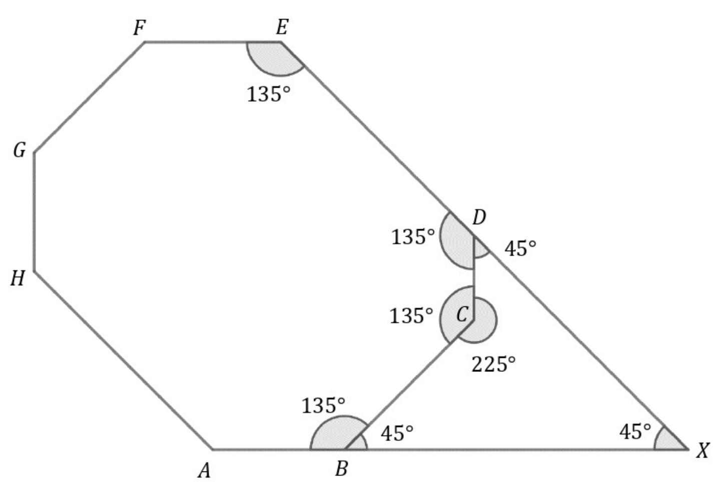
Problem 4

Given an octagon such that all its interior angles are equal, and all its sides have integer lengths.

Prove that any pair of opposite sides have equal lengths.

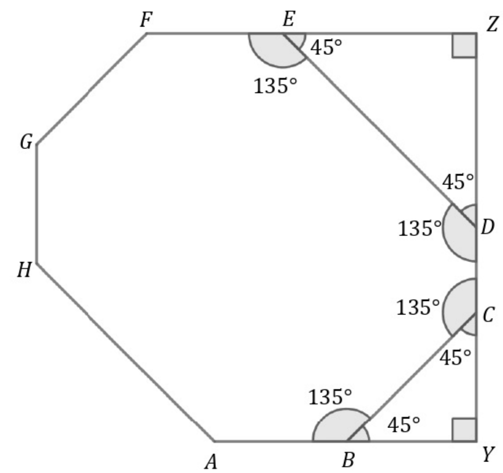
Proof. Sum of angles in an octagon is 1080°, so each interior angle is 135°. Sketch octagon $ABCDEFGH$ as shown below.

We show that each pair of opposite sides is parallel. Consider sides AB and EF. Extend sides AB and DE until they meet at point X. A simple angle chasing shows that $\angle BXE = 45°$. Since $\angle BXE + \angle XEF = 180°$, this implies that AB and DE are parallel. Similar argument can be done for other opposite sides as well.



We show that each pair of opposite sides is equal in length. Consider sides AB and EF again. Extend sides AB, EF and CD so that AB and EF meet CD at points Y and Z respectively.

A simple angle chasing shows that $\angle BYZ = \angle EZY = 90^\circ$, so CD is perpendicular to AB and EF. We calculate the height of EF from AB by using the length of YZ.



$$\begin{aligned}\text{Height} &= YZ = YC + CD + DZ = BC\sin 45^\circ + CD + DE\sin 45^\circ \\ &= \sqrt{2}(BC + DE) + CD.\end{aligned}$$

We can do the same calculation on the left side to obtain

$$\text{Height} = \sqrt{2}(FG + HA) + GH.$$

Combining both equations, we have

$$\sqrt{2}(BC + DE - FG - HA) = GH - CD.$$

If $BC + DE \neq FG + HA$, then

$$\sqrt{2} = \frac{GH - CD}{BC + DE - FG - HA}.$$

Since the sides have integer lengths, the right-hand side of the equation is rational. This is contradiction because $\sqrt{2}$ is irrational. So, $BC + DE = FG + HA$ and hence $CD = GH$. Similar argument can be done for other opposite sides as well. □

Problem 5

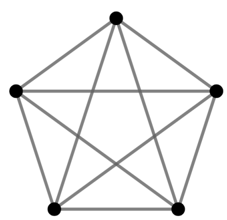
There are n guests at a gathering. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is not friends with two other guests, those two other guests cannot be friends with each other either.

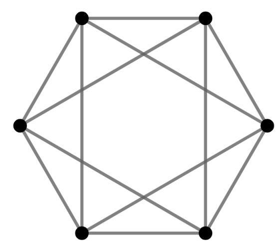
Determine all possible values of n.

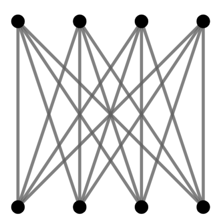
Proof. The last condition also means that if two guests befriend each other, then any other guest must befriend either one of them. First, we have to find the bounds for the number of guests. Denote the k^{th} guest as G_k. The first guest G_1 must befriend other four guests. There must be at least 5 guests, so $n \geq 5$.

Now, suppose that there are at least 9 guests. Without loss of generality, denote the friends of G_1 as G_2, G_3, G_4 and G_5. Since G_1 befriends G_2, other guests G_6, G_7, G_8 and G_9 must befriend either G_1 or G_2. Since G_1 has already four friends, these other guests must befriend with G_2. Now G_2 has five friends including G_1, which is a contradiction. There cannot be 9 guests, so $n \leq 8$.

The possible values of n are 5, 6, 7 and 8. It remains to check whether there exists configuration for each value. The cases for $n = 5, 6, 8$ are shown below.
(Note: Each vertex represents a guest, and each edge represents friendship between two guests.)







We show that $n \neq 7$. Denote the friends of G_1 as G_2, G_3, G_4 and G_5. Since G_1 befriends G_2, other guests G_6 and G_7 must befriend G_2 by using the argument above. This is also true for G_3, G_4 and G_5. Overall, G_6 and G_7 both befriend the following four guests: G_2, G_3, G_4, G_5. Now, each of G_2, G_3, G_4 and G_5 still lacks one friendship. Without loss of generality, suppose that G_2 befriends G_3. Then, G_4 must befriend either G_2 or G_3, but both have four friends already excluding G_4. This is a contradiction.

Hence, the possible values of n are 5, 6 and 8. $\square$

Problem 6

Prove that there is a positive integer m such that the number $5^{2021}m$ has no even digits (in its decimal representation).

Proof. Consider a general problem, which is $(5^n)m$. We try for a few values of n first.

$$\begin{aligned} n=1: &\quad m=1 \text{ so that } (5^1)(1)=5,\\ n=2: &\quad m=3 \text{ so that } (5^2)(3)=75,\\ n=3: &\quad m=3 \text{ so that } (5^3)(3)=375,\\ n=4: &\quad m=15 \text{ so that } (5^4)(15)=9375. \end{aligned}$$

We can guess the following: for each n, we can find m such that $(5^n)m$ has n digits, all digits are odd and the $n-1$ rightmost digits are from the previous number.

We prove the following assertion by mathematical induction: for each n, we can find m such that $(5^n)m$ has n digits and all digits are odd. The base case is clear from the example. Now, assume that there exists positive integer k such that the assertion is true with suitable m. We want to show that there exists another m' such that $(5^{k+1})m'$ has $k+1$ digits and all digits are odd.

As stated above, we want the k rightmost digits of $(5^{k+1})m'$ are from the number $(5^k)m$. This can be written as

$$(5^{k+1})m' = (10^k)d + (5^k)m \tag{1}$$

for some odd digit d. This is simplified as

$$m' = \frac{(2^k)d + m}{5}. \tag{2}$$

We have to find such m' in this form. We have to choose the correct odd digit d, so that $(2^k)d + m$ is divisible by 5. In fact, for every k and m, there is always a suitable d such that this is true. This is summarized in the table below.

2^k mod 5 \ m mod 5	0	1	2	3	4
1	$d=5$	$d=9$	$d=3$	$d=7$	$d=1$
2		$d=7$	$d=9$	$d=1$	$d=3$
3		$d=3$	$d=1$	$d=9$	$d=7$
4		$d=1$	$d=7$	$d=3$	$d=9$

To recap, given $(5^k)m$, we can find an odd digit d and then m' as (2), so that $(5^{k+1})m'$ can be written as (1). Since $(5^k)m$ has k digits and all digits are odd, this is true for $(5^{k+1})m'$ as well based on (1). Our mathematical induction is complete.

Since the assertion is true for any n, it is true for $n = 2021$ as well. □