IMONST2 2024 Primary Solutions

IMO Committee Malaysia

November 2024

§1 Problems

Problem 1. A string of letters is called good if it contains a continuous substring $IMONST$ in it. For example, the string $NSIMONSTIM$ is good, but the string $IMONNNST$ is not.
Find the number of good strings consisting of 12 letters from I, M, O, N, S, T only.

Problem 2. Jia Herng has a circle ω with center O, and P is a point outside of ω. Let PX and PY are two lines tangent to ω at X and Y, and Q is a point on segment PX. Let R is a point on the ray PY beyond Y such that $QX = RY$.
Help Jia Herng prove that the points O, P, Q, R are concyclic.

Problem 3. Janson wants to find a sequence of positive integers $a_1, a_2, \ldots, a_{2024}$ such that each term is at least 10, and a_i has exactly a_{i+1} divisors for all $1 \leq i \leq 2023$. Can you help him find one such sequence, or is this task impossible?

Problem 4. Pingu is given two positive integers m and n without any common factors greater than 1.
a) Help Pingu find positive integers p, q such that

$$\gcd(pm + q, n) \cdot \gcd(m, pn + q) = mn$$

b) Prove to Pingu that he can never find positive integers r, s such that

$$\operatorname{lcm}(rm + s, n) \cdot \operatorname{lcm}(m, rn + s) = mn$$

regardless of the choice of m and n.

Problem 5. A Duck drew a square $ABCD$, then he reflected C across B to obtain a point E. He also drew the center of the square to be F. Then, he drew a point G on ray EF beyond F such that $\angle AGC = 135°$.
Help the Duck prove that $\angle CGD = 135°$ as well.

Problem 6. There are $2n$ points on a circle, n are red and n are blue. Janson found a red frog and a blue frog at a red point and a blue point on the circle respectively. Every minute, the red frog moves to the next red point in the clockwise direction and the blue frog moves to the next blue point in the anticlockwise direction.
Prove that for any initial position of the two frogs, Janson can draw a line through the circle, such that the two frogs are always on opposite sides of the line.

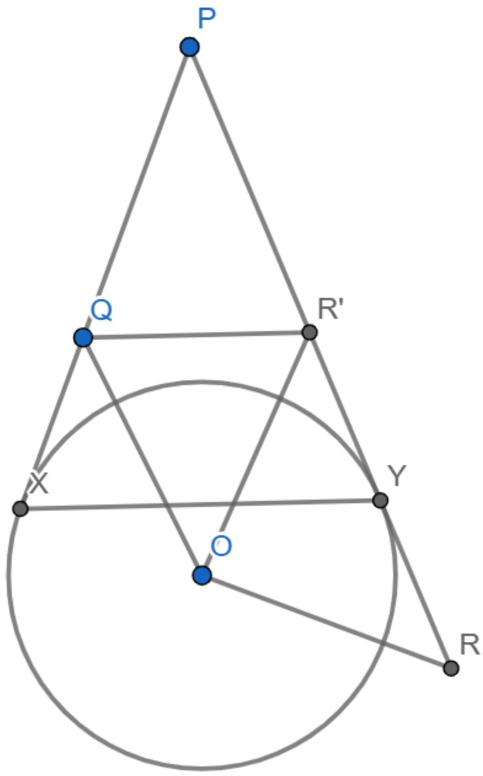
§2 Solutions

Problem 1. A string of letters is called good if it contains a continuous substring $IMONST$ in it. For example, the string $NSIMONSTIM$ is good, but the string $IMONNNST$ is not.
Find the number of good strings consisting of 12 letters from I, M, O, N, S, T only.

Solution. The continuous substring $IMONST$ can appear in $12 - 6 + 1 = 7$ different positions in a string with 12 letters. Since the other letters can be chosen freely, the number of good strings of length 12 is $7 \cdot 6^6 - 1$. This is because $IMONSTIMONST$ is the only good string that is counted twice, and all other strings can only contain at most one substring IMONST in it □

Mark Scheme:
- 1 mark for realizing 7 positions of IMONST
- 2 mark for $7 \cdot 6^6$ and forgetting the double count.
- 4 mark for realizing the double count.

Problem 2. Jia Herng has a circle ω with center O, and P is a point outside of ω. Let PX and PY are two lines tangent to ω at X and Y, and Q is a point on segment PX. Let R is a point on the ray PY beyond Y such that $QX = RY$.
Help Jia Herng prove that the points O, P, Q, R are concyclic.



Solution 1. Let R' be R reflected across Y. QR' is parallel to XY due to R' and Y being reflections of Q and X across line OP respectively. We have $\angle ORP = \angle ORY = \angle OR'Y = 180° - \angle OR'P = 180° - \angle OQP$
So, $OPQR$ is cyclic. □

Solution 2. Note that triangles OXQ and OYR are congruent, because $OX = OY$, $QX = RY$, and $\angle OXQ = \angle OYR = 90°$. Subsequently, $\angle OQX = \angle YRO$, which implies $OPQR$ is cyclic.

Mark Scheme 1:
- 3 mark for $QR' \parallel XY$
- 4 mark for conclusion.

Mark Scheme 2:
- 1 mark for stating the congruent triangles
- 3 mark for proving the congruency.
- 3 marks for the conclusion.

Problem 3. Janson wants to find a sequence of positive integers $a_1, a_2, \cdots, a_{2024}$ such that each term is at least 10, and a_i has exactly a_{i+1} divisors for all $1 \leq i \leq 2023$.
Can you help him find one such sequence, or is this task impossible?

Solution. We claim, the answer is yes. Set $a_{2024} = 10$, and $a_i = 2^{a_{i+1}-1}$ for all $i = 2023, 2022, \ldots, 1$. Since 2 is a prime, a_i has $(a_{i+1} - 1) + 1$ factors as desired. □

Mark Scheme:
- 1 mark for any related ideas of reversing the sequence
- 6 mark for correct construction.
- Deduct 1 mark if the sequence is correct but contains some terms being less than 10.

Problem 4. Pingu is given two positive integers m and n without any common factors greater than 1.
a) Help Pingu find positive integers p, q such that

$$\gcd(pm+q,n)\cdot\gcd(m,pn+q)=mn$$

b) Prove to Pingu that he can never find positive integers r, s such that

$$\operatorname{lcm}(rm+s,n)\cdot\operatorname{lcm}(m,rn+s)=mn$$

regardless of the choice of m and n.

Solution.
For part a), taking $p=q=mn$ gives $\gcd(pm+q,n)=n$ and $\gcd(m,pn+q)=m$. Hence their product is mn.

For part b), we note that $\operatorname{lcm}(a,b)\geq\max\{a,b\}$ for any positive integers a,b with equality if and only if one divides the other. Suppose such r,s exists, then we must have $\operatorname{lcm}(rm+s,n)\geq n$ and $\operatorname{lcm}(m,rn+s)\geq m$. This means that equalities holds in both inequalities, implying $rm+s\mid n$ and $rn+s\mid m$. We now have the bound

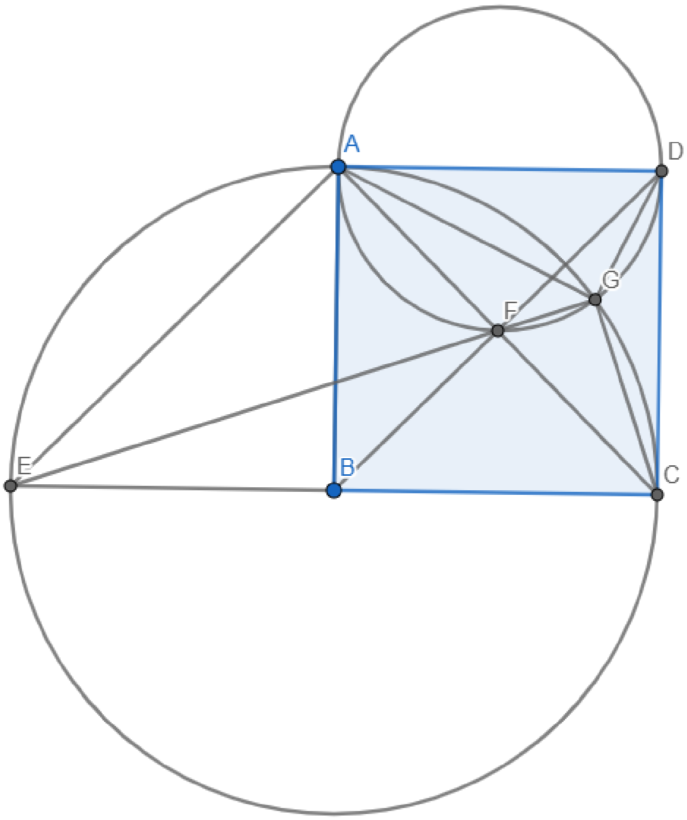
$$m+n\geq(rm+s)+(rn+s)=r(m+n)+2s\geq m+n+2$$

since $r,s\geq 1$, a contradiction. □

Mark Scheme:
- 2 marks for part a).
- 2 marks for proving $\operatorname{lcm}(rm+s,n)=n$ and related equalities.
- 3 marks for the final contradiction.

Problem 5. A Duck drew a square $ABCD$, then he reflected C across B to obtain a point E. He also drew the center of the square to be F. Then, he drew a point G on ray EF beyond F such that $\angle AGC = 135^\circ$.
Help the Duck prove that $\angle CGD = 135^\circ$ as well.

Solution.



Since E and C are symmetric on the line AB we have $\angle AEB = \angle ACB = 45^\circ$. Now since $\angle AGC = 135^\circ = 180^\circ - \angle AEC$ we have that $AECG$ is cyclic. Thus we have $\angle ADF = 45^\circ = \angle ACE = \angle AGE = \angle AGF$, so $ADGF$ is cyclic too. Now since $\angle AGD = \angle AFD = 90^\circ$ we have

$$\angle DGC = 360^\circ - \angle AGD - \angle AGC = 360^\circ - 90^\circ - 135^\circ = 135^\circ,$$

so we are done. □

Mark Scheme:
- 2 marks for $AECG$ cyclic
- 3 marks for $ADGF$ cyclic
- 2 marks for the correct conclusion.
- 7 marks for correct bash, 0 for incorrect bash (points will still be given for claims with correct proofs)

Problem 6. There are $2n$ points on a circle, n are red and n are blue. Janson found a red frog and a blue frog at a red point and a blue point on the circle respectively. Every minute, the red frog moves to the next red point in the clockwise direction and the blue frog moves to the next blue point in the anticlockwise direction.
Prove that for any initial position, Janson can draw a line through the circle, such that the two frogs are always on opposite sides of the line.

Solution 1.
Consider what happens when the two frogs move continuously from one point to the next point instead.

Graph the positions of the two frogs as they complete one revolution around the circle. The graphs intersect twice, so the frogs will cross paths at two points, then draw the line between the two points. □

Solution 2.
Consider the perspective of the red frog. As both frogs takes exactly n minutes to complete a full revolution, we claim that both frogs will cross each other exactly twice within a full revolution.

Indeed, if the blue frog passed by the red frog at least 3 times within a full revolution of the reg frog, then between the first and third time both frogs met, the blue frog would have already jumped at least a full circle, while the red frog haven't yet. This is a contradiction.

Similarly, if the blue frog only passed by the red frog once, then when the red frog completes a full revolution, the blue frog have not completed a full circle yet. This is a contradiction.

So the frogs will pass by each other exactly twice. In both occasions mark a point where both frogs crossed while passing each other, then draw the line through these two points will suffice. □

Mark Scheme:
- 2 marks for any attempt to prove the frogs will meet/pass by at two points
- 4 points for either arguing it via graphs or by contradiction.
- 1 point for the correct choice of line and the conclusion.