

IMO NATIONAL SELECTION TEST 2

SAMPLE PAPER

JUNIOR CATEGORY

Six questions · 7 marks per question · 42 marks

INSTRUCTIONS

1. This paper contains six essay questions. Answer all questions. Each question is worth 7 marks.
 2. Suggested duration: 4 hours.
 3. Show your working clearly. Marks may be awarded for correct reasoning even if the final answer is incorrect.
 4. The English questions are presented first, followed by the Bahasa Melayu questions. The solutions and suggested mark allocations follow both language sections.
 5. No calculator is required.
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ENGLISH

Each question is worth 7 marks.

Question 1

Find all positive integer pairs (x, y) with $x > y$ satisfying $x^2 - y^2 = 35$.

Question 2

Triangle ABC is isosceles, with $AB = AC$ and $\angle BAC = 20^\circ$. Point D lies strictly between A and C and satisfies $BD = BC$. Find $\angle ABD$.

Question 3

Five points are placed in a square of side length 1. Prove that two of the points are at distance at most $\frac{\sqrt{2}}{2}$.

Question 4

Find all ordered pairs of positive integers (a, b) such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

Question 5

Six distinct people, including Alice and Ben, sit around a circular table. Arrangements that differ only by rotation are considered the same, but mirror images are different. How many arrangements have Alice and Ben not sitting next to each other?

Question 6

The numbers $1, 2, 3, \dots, 2027$ are written on a board. Repeatedly choose two numbers a, b and replace them by the single number $a + b + ab$. Continue until only one number remains. How many consecutive 9s are at the end of the final number?

BAHASA MELAYU

Setiap soalan bernilai 7 markah.

Soalan 1

Cari semua pasangan integer positif (x, y) dengan $x > y$ yang memenuhi $x^2 - y^2 = 35$.

Soalan 2

Segi tiga ABC ialah segi tiga sama kaki, dengan $AB = AC$ dan $\angle BAC = 20^\circ$. Titik D terletak di antara A dengan C dan memenuhi $BD = BC$. Cari $\angle ABD$.

Soalan 3

Lima titik diletakkan di dalam sebuah segi empat sama yang panjang sisinya 1. Buktikan bahawa terdapat dua titik yang jaraknya tidak melebihi $\frac{\sqrt{2}}{2}$.

Soalan 4

Cari semua pasangan tertib integer positif (a, b) yang memenuhi $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$.

Soalan 5

Enam orang yang berlainan, termasuk Alice dan Ben, duduk mengelilingi sebuah meja bulat. Susunan yang berbeza hanya melalui putaran dianggap sama, tetapi imej cermin dianggap berbeza. Berapakah bilangan susunan apabila Alice dan Ben tidak duduk bersebelahan?

Soalan 6

Nombor 1, 2, 3, ..., 2027 ditulis pada sebuah papan. Berulang kali, pilih dua nombor a, b dan gantikan keduanya dengan satu nombor $a + b + ab$. Teruskan sehingga hanya satu nombor tinggal. Berapakah bilangan digit 9 berturut-turut pada penghujung nombor terakhir itu?

SOLUTIONS AND SUGGESTED MARK ALLOCATIONS

Each question is worth 7 marks. Accept any mathematically correct solution with clear reasoning.

Question 1

Factor $(x - y)(x + y) = 35$. The positive factor pairs with $x - y < x + y$ and the same parity are $(1, 35)$ and $(5, 7)$. They give $(x, y) = (18, 17)$ and $(6, 1)$. Thus the solutions are $\boxed{(18, 17), (6, 1)}$. **Suggested mark allocation:**

- 2 marks for factoring the difference of squares.
- 3 marks for identifying the valid factor pairs.
- 2 marks for deriving and stating both solutions.

Question 2

The base angles of ABC are 80° . Since D lies on AC , $\angle BCD = 80^\circ$. In triangle BCD , $BD = BC$, so the opposite angles are equal and $\angle BDC = 80^\circ$. Hence $\angle CBD = 20^\circ$. Therefore $\angle ABD = \angle ABC - \angle CBD = 80^\circ - 20^\circ = \boxed{60^\circ}$. **Suggested mark allocation:**

- 2 marks for the base angles of ABC .
- 2 marks for using $BD = BC$ to find $\angle CBD = 20^\circ$.
- 2 marks for relating $\angle ABD$ to $\angle ABC$.
- 1 mark for 60° .

Question 3

Divide the unit square into four smaller squares of side $1/2$. By the pigeonhole principle, at least two of the five points lie in the same smaller square. The greatest possible distance between two points in one such square is its diagonal, which is $\sqrt{(1/2)^2 + (1/2)^2} = \sqrt{2}/2$. Hence the required pair exists. **Suggested mark allocation:**

- 2 marks for dividing the square into four equal squares.
- 2 marks for applying the pigeonhole principle.
- 2 marks for calculating the small-square diagonal.
- 1 mark for the conclusion.

Question 4

Since both terms are positive, $a > 6$ and $b > 6$. Rearranging the equation gives $ab - 6a - 6b = 0$, hence $(a - 6)(b - 6) = 36$. By symmetry, assume $a \leq b$. Then $1 \leq a - 6 \leq 6$, so $a - 6$ must be a positive divisor of 36 not exceeding 6: 1, 2, 3, 4, 6. These give $(a, b) = (7, 42), (8, 24), (9, 18), (10, 15), (12, 12)$. Including the reversed pairs, all solutions are $(7, 42), (42, 7), (8, 24), (24, 8), (9, 18), (18, 9), (10, 15), (15, 10), (12, 12)$. Substitution verifies each pair. **Suggested mark allocation:**

- 1 mark for showing $a, b > 6$.
- 2 marks for obtaining $(a - 6)(b - 6) = 36$.
- 2 marks for using symmetry and listing the possible divisors.
- 1 mark for deriving the five pairs with $a \leq b$.
- 1 mark for including reversals and checking the solutions.

Question 5

There are $(6 - 1)! = 120$ circular arrangements in total. To count arrangements where Alice and Ben sit together, treat them as one block. The block and the other four people make five objects around a circle, giving $(5 - 1)!$ arrangements; Alice and Ben can switch places in 2 ways. Thus there are $2 \cdot 4! = 48$ adjacent arrangements. The required number is $120 - 48 = \boxed{72}$. **Suggested mark allocation:**

- 2 marks for the total circular arrangements.
- 3 marks for counting adjacent arrangements, including the two block orders.
- 2 marks for subtracting and obtaining 72.

Question 6

For each current number t , consider $t + 1$. Replacing a, b by $a + b + ab$ preserves the product of these shifted numbers, since $(a + b + ab) + 1 = (a + 1)(b + 1)$. Initially the product is $2 \cdot 3 \cdots 2028 = 2028!$. If the final number is F , then $F + 1 = 2028!$, so $F = 2028! - 1$. The number of trailing zeroes in $2028!$ is

$$\left\lfloor \frac{2028}{5} \right\rfloor + \left\lfloor \frac{2028}{25} \right\rfloor + \left\lfloor \frac{2028}{125} \right\rfloor + \left\lfloor \frac{2028}{625} \right\rfloor = 405 + 81 + 16 + 3 = 505.$$

There are more than 505 factors of 2, so $2028!$ ends in exactly 505 zeroes. Subtracting 1 makes these trailing zeroes into 505 consecutive 9s. Thus the answer is $\boxed{505}$. **Suggested mark allocation:**

- 2 marks for identifying and proving the shifted-product invariant.
- 1 mark for obtaining $F = 2028! - 1$.
- 2 marks for counting the factors of 5 in $2028!$.
- 1 mark for explaining why this gives exactly 505 trailing zeroes.
- 1 mark for concluding that F ends in 505 consecutive 9s.