

IMONST 2 2020–2025 Solutions

IMO COMMITTEE MALAYSIA

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Contents

1	2020	4
1.1	Primary Problem 1	4
1.2	Primary Problem 2	4
1.3	Primary Problem 3	5
1.4	Primary Problem 4, Junior Problem 1	5
1.5	Primary Problem 5, Junior Problem 2	6
1.6	Primary Problem 6, Junior Problem 3	7
1.7	Junior Problem 4, Senior Problem 1	7
1.8	Junior Problem 5, Senior Problem 2	8
1.9	Junior Problem 6, Senior Problem 3	8
1.10	Senior Problem 4	8
1.11	Senior Problem 5	9
1.12	Senior Problem 6	10
2	2021	11
2.1	Primary Problem 1	11
2.2	Primary Problem 2	11
2.3	Primary Problem 3	12
2.4	Primary Problem 4, Junior Problem 1	13
2.5	Primary Problem 5, Junior Problem 2	14
2.6	Primary Problem 6, Junior Problem 3	14
2.7	Junior Problem 4, Senior Problem 1	15
2.8	Junior Problem 5, Senior Problem 2	16
2.9	Junior Problem 6, Senior Problem 3	17
2.10	Senior Problem 4	18
2.11	Senior Problem 5	19
2.12	Senior Problem 6	20
3	2022	22
3.1	Primary Problem 1	22

3.2	Primary Problem 2	22
3.3	Primary Problem 3	22
3.4	Primary Problem 4, Junior Problem 1	24
3.5	Primary Problem 5, Junior Problem 2	24
3.6	Primary Problem 6, Junior Problem 3	25
3.7	Junior Problem 4, Senior Problem 1	25
3.8	Junior Problem 5, Senior Problem 2	26
3.9	Junior Problem 6, Senior Problem 3	26
3.10	Senior Problem 4	27
3.11	Senior Problem 5	27
3.12	Senior Problem 6	28
4	2023	29
4.1	Primary Problem 1	29
4.2	Primary Problem 2	29
4.3	Primary Problem 3	30
4.4	Primary Problem 4	30
4.5	Primary Problem 5	30
4.6	Primary Problem 6	30
4.7	Senior Problem 1	31
4.8	Senior Problem 2	31
4.9	Senior Problem 3	32
4.10	Senior Problem 4	32
4.11	Senior Problem 5	33
4.12	Senior Problem 6	33
5	2024	34
5.1	Primary Problem 1	34
5.2	Primary Problem 2	34
5.3	Primary Problem 3	35
5.4	Primary Problem 4, Junior Problem 1	35
5.5	Primary Problem 5, Junior Problem 2	35
5.6	Primary Problem 6, Junior Problem 3	36
5.7	Junior Problem 4, Senior Problem 1	37
5.8	Junior Problem 5, Senior Problem 2	37
5.9	Junior Problem 6, Senior Problem 3	38
5.10	Senior Problem 4	39
5.11	Senior Problem 5	40
5.12	Senior Problem 6	41
6	2025	44
6.1	Beginner Problem 1	44
6.2	Beginner Problem 2	44
6.3	Beginner Problem 3	44

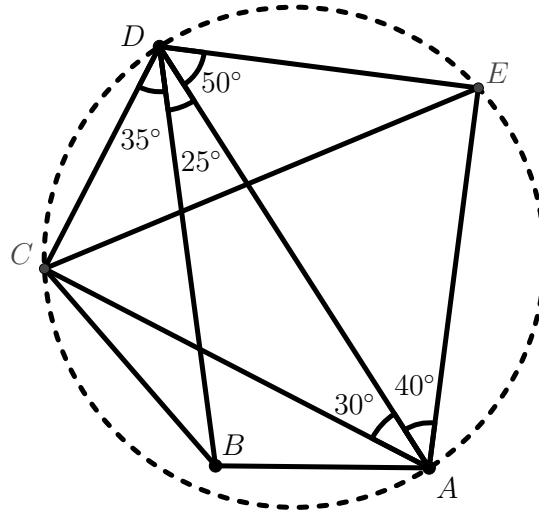
6.4	Beginner Problem 4	45
6.5	Beginner Problem 5	45
6.6	Beginner Problem 6	46
6.7	Pioneer Problem 1	46
6.8	Pioneer Problem 2	46
6.9	Pioneer Problem 3	47
6.10	Pioneer Problem 4	48
6.11	Pioneer Problem 5	49
6.12	Pioneer Problem 6	50
6.13	Junior Problem 1	50
6.14	Junior Problem 2	51
6.15	Junior Problem 3	51
6.16	Junior Problem 4	52
6.17	Junior Problem 5	53
6.18	Junior Problem 6	54
6.19	Senior Problem 1	54
6.20	Senior Problem 2	54
6.21	Senior Problem 3	55
6.22	Senior Problem 4	56
6.23	Senior Problem 5	57
6.24	Senior Problem 6	57

§1 2020

§1.1 Primary Problem 1

Given a pentagon $ABCDE$ such that $\angle CAD = 30^\circ$, $\angle DAE = 40^\circ$, $\angle ADE = 50^\circ$, $\angle ADB = 25^\circ$, and $\angle BDC = 35^\circ$. Find the sum of angles $\angle ACE + \angle CED$.

Solution.



Observe that

$$\begin{aligned}\angle CDE + \angle CAE &= (\angle ADE + \angle ADB + \angle BDC) + (\angle CAD + \angle DAE) \\ &= (50^\circ + 25^\circ + 35^\circ) + (30^\circ + 40^\circ) \\ &= 180^\circ,\end{aligned}$$

which means that $ACDE$ is a cyclic quadrilateral. Therefore,

$$\begin{aligned}\angle ACE + \angle CED &= \angle ADE + \angle CAD \\ &= 50^\circ + 30^\circ \\ &= 80^\circ.\end{aligned}$$

§1.2 Primary Problem 2

Prove that the sum of the first n odd positive integers is equal to n^2 , for all positive integers n .

Solution. The sum of an arithmetic sequence with n terms is

$$S = \frac{n[2a + (n-1)d]}{2},$$

where a is the first term and d is the common difference. For the first n odd positive integers, $a = 1$ and $d = 2$, so

$$\begin{aligned} S &= \frac{n[2(1) + (n-1)(2)]}{2} \\ &= \frac{n(2n)}{2} \\ &= n^2. \end{aligned}$$

§1.3 Primary Problem 3

A positive integer N has four digits. The sum of the digits of N is equal to $2020 - N$. Find all possible values of N .

Solution. Let $N = 1000a + 100b + 10c + d$, where a, b, c, d are integers with $1 \leq a \leq 9$ and $0 \leq b, c, d \leq 9$. Then

$$\begin{aligned} 2020 - (1000a + 100b + 10c + d) &= a + b + c + d \\ 1001a + 101b + 11c + 2d &= 2020. \end{aligned}$$

Since $1001a \leq 2020$, we have $a \in \{1, 2\}$.

If $a = 2$, then $101b + 11c + 2d = 18$, so $b = 0$ and $11c + 2d = 18$. Then c is even and $c \leq 1$, so $c = 0$ and $d = 9$, giving $N = 2009$.

If $a = 1$, then $101b + 11c + 2d = 1019$. Since $11c + 2d \leq 99 + 18 = 117$, we need $101b \geq 902$, so $b = 9$ and

$$11c + 2d = 110.$$

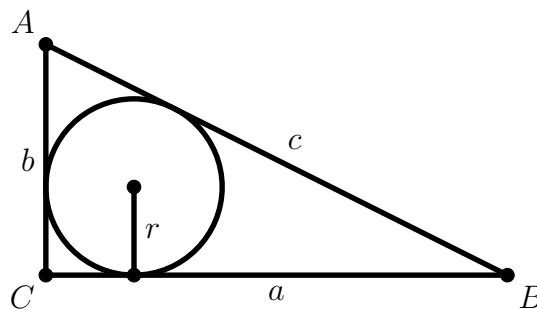
Then c is even, so $c \leq 8$ and $2d \geq 110 - 88 = 22$, which contradicts $d \leq 9$.

Therefore, the only possible value is $N = 2009$.

§1.4 Primary Problem 4, Junior Problem 1

A right triangle has sides of lengths a , b , and c (c is the length of the hypotenuse). Inside the triangle, we place a circle such that it touches all three sides of the triangle. Prove that the radius of this circle is equal to $\frac{a + b - c}{2}$.

Solution.



The area of a triangle is rs , where r is the inradius and s is the semiperimeter. Since the legs of the right triangle are a and b ,

$$\frac{1}{2}ab = \frac{a+b+c}{2} \cdot r.$$

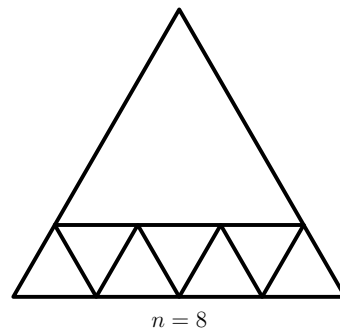
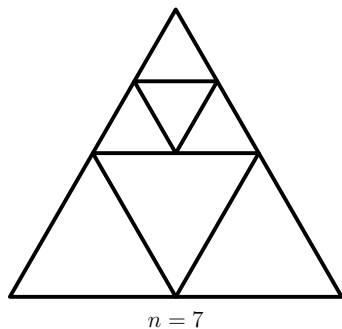
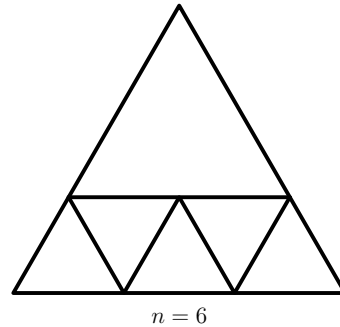
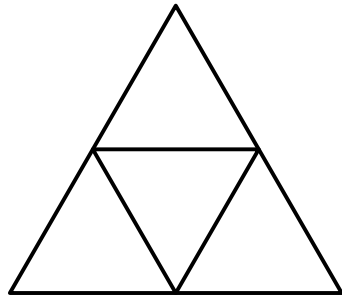
By Pythagoras' theorem, $a^2 + b^2 = c^2$. Therefore,

$$\begin{aligned} r &= \frac{ab}{a+b+c} \\ &= \frac{(a+b)^2 - (a^2 + b^2)}{2(a+b+c)} \\ &= \frac{(a+b)^2 - c^2}{2(a+b+c)} \\ &= \frac{(a+b+c)(a+b-c)}{2(a+b+c)} \\ &= \frac{a+b-c}{2}. \end{aligned}$$

§1.5 Primary Problem 5, Junior Problem 2

Prove that for any integer $n \geq 6$, we can divide an equilateral triangle completely into n smaller equilateral triangles.

Solution.



Since subdividing any one equilateral triangle into 4 increases the total number of triangles by 3, starting from 6, 7, 8 covers all integers modulo 3. Therefore we just need to show that if we can divide it to 6, 7, or 8 triangles we can get all possible triangles.

§1.6 Primary Problem 6, Junior Problem 3

Given integers a and b such that $a^2 + b^2$ is divisible by 11. Prove that a and b are both divisible by 11.

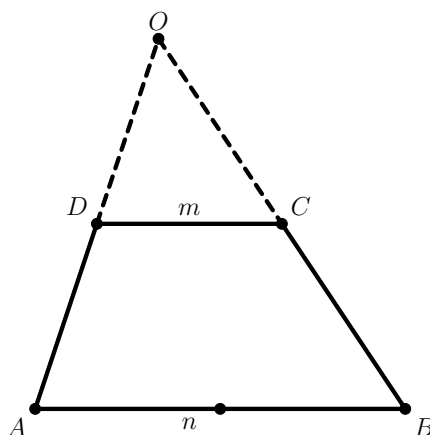
Solution. The residues of perfect squares modulo 11 are 0, 1, 4, 9, 5, 3. There is no possible way to form a multiple of 11 by summing 2 of them other than choosing two 0's which means the numbers we choose are multiples of 11. Therefore both numbers must be divisible by 11.

§1.7 Junior Problem 4, Senior Problem 1

Given a trapezium with two parallel sides of lengths m and n , where m and n are integers. Prove that it is possible to divide the trapezium into several congruent triangles.

Note: Two triangles are congruent if they have the same shape and the same size.

Solution.



WLOG let $m < n$ as $m = n$ is trivial by just drawing a diagonal and we're done. Let O be the intersection of AD and BC . Since $DC \parallel AB$, $\triangle ODC$ is similar to $\triangle OAB$ with scale factor $\frac{DC}{AB} = \frac{m}{n}$.

Divide each of OA and OB into n equal parts and draw lines through the division points parallel to the sides of $\triangle OAB$. This tiles $\triangle OAB$ with n^2 congruent triangles, each of which is $\triangle OAB$ scaled down by the factor $\frac{1}{n}$. Since $OD = \frac{m}{n} \cdot OA$ and $OC = \frac{m}{n} \cdot OB$, the segment DC is one of the lines of this grid, so $\triangle ODC$ consists of exactly m^2 of those same congruent triangles.

Therefore, removing these m^2 tiles from $\triangle OAB$ leaves the trapezium $ABCD$ divided into $n^2 - m^2$ congruent triangles.

§1.8 Junior Problem 5, Senior Problem 2

Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2019} - \frac{1}{2020} = \frac{1}{1011} + \frac{1}{1012} + \frac{1}{1013} + \cdots + \frac{1}{2020}.$$

Solution. Observe that the alternating sum is the sum of all the terms minus twice the even terms $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2020}$. Therefore,

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2019} - \frac{1}{2020} &= \sum_{n=1}^{2020} \frac{1}{n} - 2 \sum_{n=1}^{1010} \frac{1}{2n} \\ &= \sum_{n=1}^{2020} \frac{1}{n} - \sum_{n=1}^{1010} \frac{1}{n} \\ &= \sum_{n=1011}^{2020} \frac{1}{n} \\ &= \frac{1}{1011} + \frac{1}{1012} + \cdots + \frac{1}{2020}. \end{aligned}$$

§1.9 Junior Problem 6, Senior Problem 3

Find all possible integer values of n such that $12n^2 + 12n + 11$ is a 4-digit number with all 4 digits equal.

Solution. Given that $12n^2 + 12n + 11 = 1111k$, where k is the repeated digit, so $1 \leq k \leq 9$.

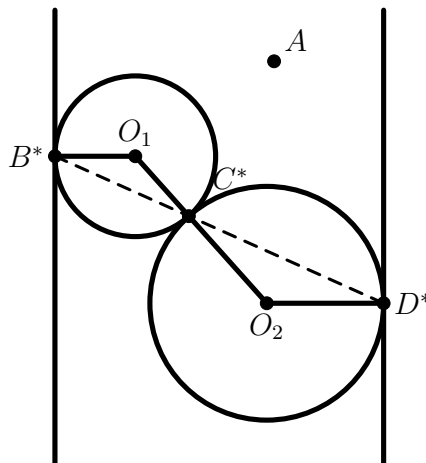
$k \equiv 1 \pmod{4}$ $k \equiv 2 \pmod{3}$ Therefore k must be 5. So by solving $12n^2 + 12n + 11 = 5555$
 $n = 21, -22$

§1.10 Senior Problem 4

Given four circles $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 . Circles Γ_1 and Γ_2 are externally tangent at point A . Circles Γ_2 and Γ_3 are externally tangent at point B . Circles Γ_3 and Γ_4 are externally tangent at point C . Circles Γ_4 and Γ_1 are externally tangent at point D .

Prove that quadrilateral $ABCD$ is cyclic.

Solution. Consider an inversion around the point A , and let B^*, C^*, D^* be the images of B, C, D . Since Γ_1 and Γ_2 pass through A and are tangent there, they are mapped to two parallel lines. The circles Γ_3 and Γ_4 do not pass through A , so they are mapped to two circles, which are tangent to each other at C^* . Moreover, the image of Γ_3 is tangent to the image of Γ_2 at B^* , and the image of Γ_4 is tangent to the image of Γ_1 at D^* . We wish to show that B^*, C^*, D^* are collinear. Our inversion gives the following image, consisting of two parallel lines and two tangent circles.



Let O_1, O_2 be the centers of the two circles in the image, such that B^* lies on the circle with center O_1 and D^* lies on the circle with center O_2 . We know that O_1, C^*, O_2 are collinear. Moreover, we have $B^*O_1 = C^*O_1$ and $D^*O_2 = C^*O_2$. Finally, since $\overline{B^*O_1} \parallel \overline{D^*O_2}$ (both are perpendicular to the parallel lines), we have that $\angle B^*O_1C^* = \angle C^*O_2D^*$. Therefore, the isosceles triangles $B^*O_1C^*$ and $C^*O_2D^*$ are similar, so $\angle O_1C^*B^* = \angle O_2C^*D^*$. Since O_1, C^*, O_2 are collinear, these are vertical angles, and it follows that B^*, C^*, D^* are collinear.

Finally, the line through B^*, C^*, D^* does not pass through A , so inverting back maps it to a circle through A . This circle passes through B, C and D , hence $ABCD$ is cyclic, as desired.

§1.11 Senior Problem 5

Let p and q be real numbers such that the quadratic equation $x^2 + px + q = 0$ has two distinct real solutions x_1 and x_2 . Suppose that $|x_1 - x_2| = 1$ and $|p - q| = 1$. Prove that all of p, q, x_1 , and x_2 are integers.

Solution. By the quadratic formula, $x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$, so the difference between the solutions is

$$|x_1 - x_2| = \sqrt{p^2 - 4q} = 1 \implies p^2 - 4q = 1.$$

Since $|p - q| = 1$, we have $q = p - 1$ or $q = p + 1$.

If $q = p - 1$, then

$$p^2 - 4(p - 1) = 1$$

$$p^2 - 4p + 3 = 0$$

$$(p - 1)(p - 3) = 0,$$

so $(p, q) = (1, 0)$ or $(3, 2)$.

If $q = p + 1$, then

$$\begin{aligned} p^2 - 4(p + 1) &= 1 \\ p^2 - 4p - 5 &= 0 \\ (p + 1)(p - 5) &= 0, \end{aligned}$$

so $(p, q) = (-1, 0)$ or $(5, 6)$.

In every case p and q are integers, and p is odd. Since $\sqrt{p^2 - 4q} = 1$,

$$x_{1,2} = \frac{-p \pm 1}{2},$$

and both $-p + 1$ and $-p - 1$ are even, so x_1 and x_2 are integers too. Explicitly:

p	q	x_1	x_2
1	0	0	-1
3	2	-1	-2
-1	0	1	0
5	6	-2	-3

Therefore p , q , x_1 and x_2 are all integers.

§1.12 Senior Problem 6

Consider the following one-person game: A player starts with score 0 and writes the number 20 on an empty whiteboard. At each step, she may erase any one integer (call it a) and writes two positive integers (call them b and c) such that $b + c = a$. The player then adds $b \times c$ to her score. She repeats the step several times until she ends up with all 1s on the whiteboard. Then the game is over, and the final score is calculated.

Example: At the first step, a player erases 20 and writes 14 and 6, and gets a score of $14 \times 6 = 84$. In the next step, she erases 14, writes 9 and 5, and adds $9 \times 5 = 45$ to her score. Her score is now $84 + 45 = 129$. In the next step, she may erase any of the remaining numbers on the whiteboard: 5, 6, or 9. She continues until the game is over.

Alya and Bob play the game separately. Alya manages to get the highest possible final score. Bob, however, manages to get the lowest possible final score. What is the difference between Alya's and Bob's final scores?

Solution. The final score difference is 0. Given a number $n \geq 2$, we will get a total score of $\frac{n(n-1)}{2}$ for base case $n = 2$ is trivial. Assume that for each $n = 1, 2, \dots, k$, the total score is $\frac{n(n-1)}{2}$. For $n = k + 1$, the only valid move to make is erasing $k + 1$ and splitting it to j and $k + 1 - j$, where $j \in \{1, 2, \dots, k\}$. By induction hypothesis, the score will be

$j(k + 1 - j) + \frac{(k + 1 - j)(k - j)}{2} + \frac{(j)(j - 1)}{2} = \frac{(k + 1)(k)}{2}$ which by strong induction we get that the score is constant.

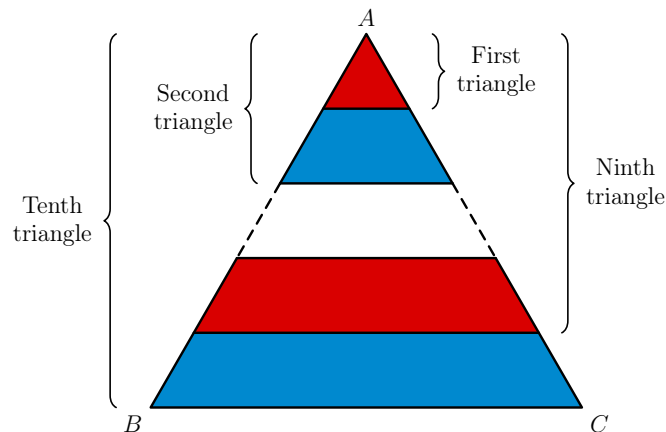
§2 2021

§2.1 Primary Problem 1

An equilateral triangle ABC is divided by nine lines parallel to BC into ten bands that are equally wide. We colour the bands alternately red and blue, with the smallest band coloured red. The difference between the total area in red and the total area in blue is 20 cm^2 .

What is the area of triangle ABC ?

Solution. Consider the following diagram. Denote the area of the first triangle as x .



Since each band has equal wide, the height of the second triangle is twice the height of the first one, and the same is true for its base too. By calculation, the area of the second triangle is $4x$. Similarly, the height and base of the third triangle are thrice the height and base of the first one, so its area is $9x$. Continuing this way, the areas of the triangles in this order are given as follows: $x, 4x, 9x, 16x, \dots, 81x, 100x$.

Now, the second band is obtained by removing the first triangle from the second one. So, the area of the second band is $3x$. Similarly, the third band is obtained by removing the second triangle from the third one, so its area is $5x$. Continuing this way, the areas of the bands in this order are given as follows: $x, 3x, 5x, 7x, \dots, 17x, 19x$.

Overall, the areas of the red bands are $x, 5x, 9x, 13x$ and $17x$, with total area $45x$. The areas of the blue bands are $3x, 7x, 11x, 15x$ and $19x$, with total area $55x$. The difference between the total areas is $10x$. It is given that this difference is 20 cm^2 , so $x = 2 \text{ cm}^2$.

The area of triangle ABC is $100x$, which is 200 cm^2 .

§2.2 Primary Problem 2

The five numbers a, b, c, d and e satisfy the inequalities

$$a + b < c + d < e + a < b + c < d + e.$$

Among the five numbers, which one is the smallest, and which one is the largest?

Solution. From the inequalities, we pair up expressions containing the same number. We have

- (i) $a + b < e + a$ and so $b < e$,
- (ii) $a + b < b + c$ and so $a < c$,
- (iii) $c + d < b + c$ and so $d < b$,
- (iv) $c + d < d + e$ and so $c < e$, and
- (v) $e + a < d + e$ and so $a < d$

by cancelling the same number in each inequality. Combining these inequalities, we obtain

$$a < c < e \quad \text{and} \quad a < d < b < e.$$

So, the smallest number is a , and the largest number is e .

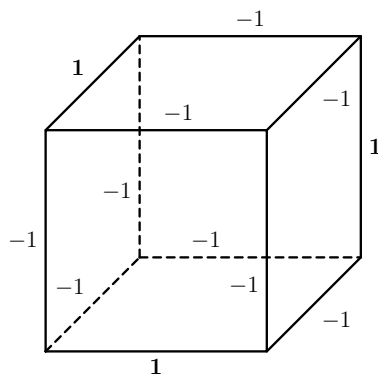
§2.3 Primary Problem 3

Given a cube. On each edge of the cube, we write a number, either 1 or -1 . For each face of the cube, we multiply the four numbers on the edges of this face, and write the product on this face. Finally, we add all the eighteen numbers that we wrote down on the edges and faces of the cube.

What is the smallest possible sum that we can get?

Solution. Note that a face has number -1 if and only if exactly one or three of its edges have number -1 . For the smallest possible sum, the idea is to increase the number of -1 's and reduce the number of 1's on the cube. We can check each possible configuration as follows:

- (i) eighteen -1 's and no 1. This means that all edges and faces have number -1 . However, if all edges have number -1 , then all faces have number 1. This is a contradiction.
- (ii) seventeen -1 's and one 1. If the number 1 is on an edge, then any face without that edge will have number 1. If it is on a face, then all edges have number -1 and thus, other faces have number 1. Both cases contradict that there is only one 1.
- (iii) sixteen -1 's and two 1's. If the two 1's are both on edges, then any face without these edges will have number 1. If they are both on faces, then all edges have number -1 and thus, other faces have number 1. If they are on an edge and a face, then any other face without that edge will have number 1. All cases contradict that there are only two 1's.
- (iv) fifteen -1 's and three 1's. This is possible by setting three 1's on certain edges as shown below. The sum is -12 .



Since it is impossible to go lower as explained above, the smallest possible sum is -12 .

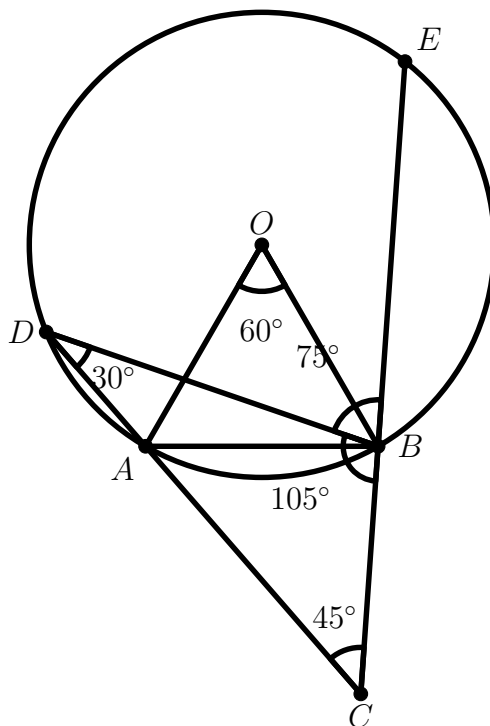
§2.4 Primary Problem 4, Junior Problem 1

Given a circle with center O . Points A and B lie on the circle such that triangle OBA is equilateral. Let C be a point outside the circle with $\angle ACB = 45^\circ$. Line CA intersects the circle at point D , and line CB intersects the circle at point E .

Find $\angle DBE$.

Solution. Sketch the circle with center O and the triangle OAB such that the side AB is below O . Since OAB is an equilateral triangle, we know $\angle AOB = 60^\circ$. Any angle subtended by the arc AB at the circumference is half of that at the center, which is 30° . Since $\angle ACB = 45^\circ$ which is bigger than 30° , point C must be somewhere below the line AB . We continue to sketch the remaining points and lines as shown below.

Angle $\angle ADB$ is subtended by the arc AB at the circumference, so $\angle ADB = 30^\circ$. Since the sum of angles in any triangle including BCD is 180° , we have $\angle CBD = 105^\circ$. Since CBE is a straight line, we have $\angle CBD + \angle DBE = 180^\circ$, and thus $\angle DBE = 75^\circ$.



§2.5 Primary Problem 5, Junior Problem 2

Can we find positive integers a and b such that both $a^2 + b$ and $b^2 + a$ are perfect squares?

Solution. Suppose that such a and b exist. The perfect square $a^2 + b$ is larger than the perfect square a^2 . The next perfect square after a^2 is $(a+1)^2$. So, $a^2 + b$ is either $(a+1)^2$ itself or more. This implies $a^2 + b \geq (a+1)^2$. The previous inequality is simplified to be $b \geq 2a + 1$.

We argue similarly to obtain $b^2 + a \geq (b+1)^2$ and thus $a \geq 2b + 1$. Adding both inequalities, we have $a + b \geq 2a + 2b + 2$, or equivalently, $a + b \leq -2$. This contradicts the fact that a and b are positive and so is their sum $a + b$.

So, such positive integers a and b cannot exist.

§2.6 Primary Problem 6, Junior Problem 3

Given a sequence of positive integers

$$a_1, a_2, a_3, a_4, a_5, \dots$$

such that $a_2 > a_1$ and $a_{n+2} = 3a_{n+1} - 2a_n$ for all $n \geq 1$.

Prove that $a_{2021} > 2^{2019}$.

Solution. Equation $a_{n+2} = 3a_{n+1} - 2a_n$ can be written as $a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n)$.

Using the latter, we can write the terms as follows:

$$\begin{aligned}
 n = 1 : \quad & a_3 - a_2 = 2(a_2 - a_1), \\
 n = 2 : \quad & a_4 - a_3 = 2(a_3 - a_2), \\
 n = 3 : \quad & a_5 - a_4 = 2(a_4 - a_3), \\
 & \dots \\
 n = 2019 : \quad & a_{2021} - a_{2020} = 2(a_{2020} - a_{2019}).
 \end{aligned}$$

We can do consecutive substitution as follows:

$$\begin{aligned}
 a_{2021} - a_{2020} &= 2(a_{2020} - a_{2019}) \\
 &= 2^2(a_{2019} - a_{2018}) \\
 &= 2^3(a_{2018} - a_{2017}) \\
 &= \dots \\
 &= 2^{2019}(a_2 - a_1).
 \end{aligned}$$

Overall, we have

$$a_{2021} = 2^{2019}(a_2 - a_1) + a_{2020}.$$

Now, a_1 and a_2 are positive integers such that $a_2 > a_1$. This implies that a_2 must either be $a_1 + 1$ or more. In other words, $a_2 \geq a_1 + 1$ or equivalently, $a_2 - a_1 \geq 1$. Since a_{2020} is positive, we have $a_{2020} > 0$. Using the last two inequalities on the equation above, we deduce that $a_{2021} > 2^{2019}$.

§2.7 Junior Problem 4, Senior Problem 1

Find all values of n such that there exists a rectangle with integer side lengths, perimeter n , and area $2n$.

Solution. Denote a and b as the length and width of such rectangle. Calculations on perimeter and area of the rectangle give us

$$2a + 2b = n \quad \text{and} \quad ab = 2n$$

which then implies

$$ab - 4a - 4b = 0.$$

We do factorisation by applying the following trick:

$$\begin{aligned}
 ab - 4a - 4b + 16 &= 16 \\
 \implies (a - 4)(b - 4) &= 16.
 \end{aligned}$$

Note that a and b are integers, and so are $a - 4$ and $b - 4$. Both expressions are factors of 16, which can be positive or negative factors. The possible pairs of values for $(a - 4, b - 4)$ are as follows: $(1, 16)$, $(2, 8)$, $(4, 4)$, $(8, 2)$, $(16, 1)$, $(-1, -16)$, $(-2, -8)$, $(-4, -4)$, $(-8, -2)$, $(-16, -1)$. Solving for each pair, we have the positive integer for (a, b) as follows: $(5, 20)$, $(6, 12)$, $(8, 8)$, $(12, 6)$, $(20, 5)$. Hence, the possible values of n are 32, 36 and 50.

§2.8 Junior Problem 5, Senior Problem 2

Six teams participate in a hockey tournament. Each team plays once against every other team. In each game, a team is awarded 3 points for a win, 1 point for a draw, and 0 point for a loss.

After the tournament, the teams are ranked by total points. No two teams have the same total points. Each team (except the bottom team) has 2 points more than the team ranking one place lower.

Prove that the team that finished fourth has won two games and lost three games.

Solution. Denote T_1 , T_2 and up till T_6 as the team ranking first, second and so on respectively. Denote p as the total points of the last team T_6 . Based on the given conditions, the sum of points for all teams is $6p + 30$.

A game can end up with a win-lose which contributes a total 3 points to the sum, or a draw which contributes a total 2 points to the sum. Since there are 15 games, the maximum sum is 45 which is achieved when each game ends up with win-lose. So, we have $6p + 30 \leq 45$ which then implies that $p \leq 2$.

We check which values of p is possible.

(a) $p = 0$

Based on the conditions, T_5 has 2 points. Note that T_6 loses in each game, including against T_5 . This implies that T_5 has at least 3 points, which is a contradiction.

(b) $p = 1$

The sum of points for all teams is 36. Let a and b be the number of games ending up with win-lose and draw respectively. Based on the number of games and also point contribution, we have $a + b = 15$ and $3a + 2b = 36$. This implies that $a = 6$ and $b = 9$. We focus on the games ending up with win-lose, in which there shall be 6 such games.

Based on points, T_6 has a draw and four losses. T_5 has either a win and four losses, or three draws and two losses. In the first case for T_5 , the win must come from the game against T_6 . Overall in this case, the following games end up with win-lose: T_6 against each team except T_5 , and T_5 against each team except T_6 .

In the second case for T_5 , a draw must come from the game against T_6 . So, T_6 loses against each team other than T_5 , and that includes T_4 . Since T_4 has 5 points, it must end up with a win, two draws and two losses. Consequently for T_5 , another draw must come from the game against T_4 . Overall in this case, the following games end up with win-lose: T_6 against each team except T_5 , two games in which T_5 loses (that are not against T_6 and T_4) and two games in which T_4 loses (that are not against T_6 and T_5).

In both cases, there are at least 8 games ending up with win-lose in the tournament. This contradicts our earlier calculation.

(c) $p = 2$

Using similar calculation in (ii), there are 12 games and 3 games ending up with

win-lose and draw respectively in the tournament. We focus on the games ending up with draw, in which there shall be 3 such games.

Based on the points, T_6 has exactly two draws, T_5 and T_2 each has at least one draw and T_3 has at least two draws. So, those three games which end up with draw must involve these teams only. Since T_4 has 6 points and does not have any draw, it must have two wins and three losses.

Overall, we deduce that

- (i) T_1 has four wins and a loss,
- (ii) T_2 has three wins, a draw and a loss,
- (iii) T_3 has two wins, two draws and a loss,
- (iv) T_4 has two wins and three losses,
- (v) T_5 has a win, a draw and three losses, and
- (vi) T_6 has two draws and three losses.

We have to check whether this outcome is possible in the tournament. Indeed, one such example is shown below.

Team	Wins against	Draws against	Loses against
T_1	T_2, T_3, T_4, T_6	None	T_5
T_2	T_4, T_5, T_6	T_3	T_1
T_3	T_4, T_5	T_2, T_6	T_1
T_4	T_5, T_6	None	T_1, T_2, T_3
T_5	T_1	T_6	T_2, T_3, T_4
T_6	None	T_3, T_5	T_1, T_2, T_4

We have proved that every possible outcome of the tournament always leads to T_4 having two wins and three losses.

§2.9 Junior Problem 6, Senior Problem 3

Let x and y be two rational numbers such that

$$x^5 + y^5 = 2x^2y^2.$$

Prove that $\sqrt{1 - xy}$ is also a rational number.

Note: A rational number is a number that can be expressed as $\frac{a}{b}$ where a and b are integers ($b \neq 0$).

Solution. If either $x = 0$ and $y = 0$, then we have $\sqrt{1 - xy} = 1$ which is clearly rational. To avoid such trivial case, we assume that x and y are non-zero. Observe the following

trick:

$$\begin{aligned}
 x^5 + y^5 = 2x^2y^2 &\implies (x^5 + y^5)^2 = 4x^4y^4 \\
 &\implies x^{10} + y^{10} + 2x^5y^5 = 4x^4y^4 \\
 &\implies (x^5 - y^5)^2 + 4x^5y^5 = 4x^4y^4 \\
 &\implies (x^5 - y^5)^2 = 4x^4y^4(1 - xy) \\
 &\implies 1 - xy = \frac{(x^5 - y^5)^2}{4x^4y^4} \\
 &\implies \sqrt{1 - xy} = \frac{|x^5 - y^5|}{2x^2y^2}.
 \end{aligned}$$

Since x and y are rational and the arithmetic operations preserve rational property of numbers, the right-hand side of the equation is rational, and so is $\sqrt{1 - xy}$.

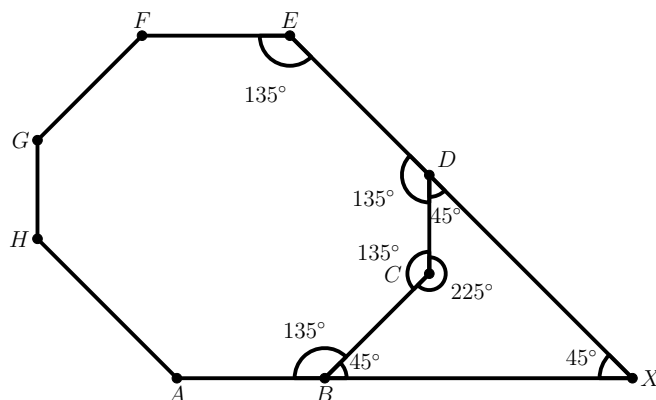
§2.10 Senior Problem 4

Given an octagon such that all its interior angles are equal, and all its sides have integer lengths.

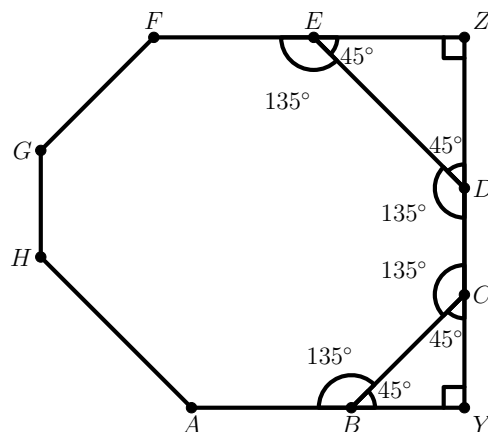
Prove that any pair of opposite sides have equal lengths.

Solution. Sum of angles in an octagon is 1080° , so each interior angle is 135° . Sketch octagon $ABCDEFGH$ as shown below.

We show that each pair of opposite sides is parallel. Consider sides AB and EF . Extend sides AB and DE until they meet at point X . A simple angle chasing shows that $\angle BXE = 45^\circ$. Since $\angle BXE + \angle XEF = 180^\circ$, this implies that AB and EF are parallel. Similar argument can be done for other opposite sides as well.



We show that each pair of opposite sides is equal in length. Consider sides AB and EF again. Extend sides AB , EF and CD so that AB and EF meet CD at points Y and Z respectively. A simple angle chasing shows that $\angle BYZ = \angle EZY = 90^\circ$, so CD is perpendicular to AB and EF . We calculate the height of EF from AB by using the length of YZ .



$$\begin{aligned}\text{Height} &= YZ = YC + CD + DZ = BC \sin 45^\circ + CD + DE \sin 45^\circ \\ &= \frac{\sqrt{2}}{2}(BC + DE) + CD.\end{aligned}$$

We can do the same calculation on the left side to obtain

$$\text{Height} = \frac{\sqrt{2}}{2}(FG + HA) + GH.$$

Combining both equations, we have

$$\frac{\sqrt{2}}{2}(BC + DE - FG - HA) = GH - CD.$$

If $BC + DE \neq FG + HA$, then

$$\frac{\sqrt{2}}{2} = \frac{GH - CD}{BC + DE - FG - HA}.$$

Since the sides have integer lengths, the right-hand side of the equation is rational. This is a contradiction because $\sqrt{2}$ is irrational. So, $BC + DE = FG + HA$ and hence $CD = GH$. Similar argument can be done for other opposite sides as well.

§2.11 Senior Problem 5

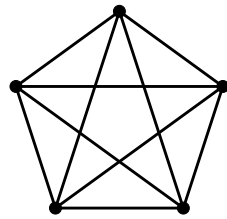
There are n guests at a gathering. Any two guests are either friends or not friends. Every guest is friends with exactly four of the other guests. Whenever a guest is not friends with two other guests, those two other guests cannot be friends with each other either. Determine all possible values of n .

Solution. The last condition also means that if two guests befriend each other, then any other guest must befriend either one of them. First, we have to find the bounds for the number of guests. Denote the k^{th} guest as G_k . The first guest G_1 must befriend other four guests. There must be at least 5 guests, so $n \geq 5$.

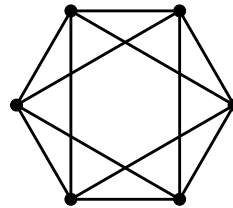
Now, suppose that there are at least 9 guests. Without loss of generality, denote the friends of G_1 as G_2, G_3, G_4 and G_5 . Since G_1 befriends G_2 , other guests G_6, G_7, G_8 and G_9 must befriend either G_1 or G_2 . Since G_1 has already four friends, these other guests must befriend with G_2 . Now G_2 has five friends including G_1 , which is a contradiction. There cannot be 9 guests, so $n \leq 8$.

The possible values of n are 5, 6, 7 and 8. It remains to check whether there exists configuration for each value. The cases for $n = 5, 6, 8$ are shown below.

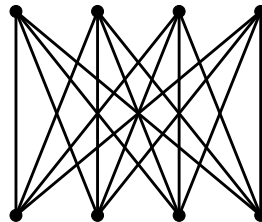
(Note: Each vertex represents a guest, and each edge represents friendship between two guests.)



$n = 5$



$n = 6$



$n = 8$

We show that $n \neq 7$. Denote the friends of G_1 as G_2, G_3, G_4 and G_5 . Since G_1 befriends G_2 , other guests G_6 and G_7 must befriend G_2 by using the argument above. This is also true for G_3, G_4 and G_5 . Overall, G_6 and G_7 both befriend the following four guests: G_2, G_3, G_4, G_5 . Now, each of G_2, G_3, G_4 and G_5 still lacks one friendship. Without loss of generality, suppose that G_2 befriends G_3 . Then, G_4 must befriend either G_2 or G_3 , but both have four friends already excluding G_4 . This is a contradiction.

Hence, the possible values of n are 5, 6 and 8.

§2.12 Senior Problem 6

Prove that there is a positive integer m such that the number $5^{2021}m$ has no even digits (in its decimal representation).

Solution. Consider a general problem, which is $(5^n)m$. We try for a few values of n

first.

$$\begin{aligned}
 n = 1 : \quad m &= 1 \text{ so that } (5^1)(1) = 5, \\
 n = 2 : \quad m &= 3 \text{ so that } (5^2)(3) = 75, \\
 n = 3 : \quad m &= 3 \text{ so that } (5^3)(3) = 375, \\
 n = 4 : \quad m &= 15 \text{ so that } (5^4)(15) = 9375.
 \end{aligned}$$

We can guess the following: for each n , we can find m such that $(5^n)m$ has n digits, all digits are odd and the $n - 1$ rightmost digits are from the previous number.

We prove the following assertion by mathematical induction: for each n , we can find m such that $(5^n)m$ has n digits and all digits are odd. The base case is clear from the example. Now, assume that there exists positive integer k such that the assertion is true with suitable m . We want to show that there exists another m' such that $(5^{k+1})m'$ has $k + 1$ digits and all digits are odd.

As stated above, we want the k rightmost digits of $(5^{k+1})m'$ are from the number $(5^k)m$. This can be written as

$$(5^{k+1})m' = (10^k)d + (5^k)m \quad (1)$$

for some odd digit d . This is simplified as

$$m' = \frac{(2^k)d + m}{5}. \quad (2)$$

We have to find such m' in this form. We have to choose the correct odd digit d , so that $(2^k)d + m$ is divisible by 5. In fact, for every k and m , there is always a suitable d such that this is true. This is summarized in the table below.

$\begin{matrix} m \bmod 5 \\ \backslash \\ 2^k \bmod 5 \end{matrix}$	0	1	2	3	4
1	$d = 5$	$d = 9$	$d = 3$	$d = 7$	$d = 1$
2		$d = 7$	$d = 9$	$d = 1$	$d = 3$
3		$d = 3$	$d = 1$	$d = 9$	$d = 7$
4		$d = 1$	$d = 7$	$d = 3$	$d = 9$

To recap, given $(5^k)m$, we can find an odd digit d and then m' as (2), so that $(5^{k+1})m'$ can be written as (1). Since $(5^k)m$ has k digits and all digits are odd, this is true for $(5^{k+1})m'$ as well based on (1). Our mathematical induction is complete.

Since the assertion is true for any n , it is true for $n = 2021$ as well.

§3 2022

§3.1 Primary Problem 1

Given a polygon $ABCDEFGHIJ$.

How many diagonals does the polygon have?

Solution. Note that we may draw $\binom{10}{2} = 45$ line segments using two vertices as endpoints, but 10 of those are the sides of the polygon, hence the number of diagonals is

$$\binom{10}{2} - 10 = 35.$$

□

§3.2 Primary Problem 2

The following list shows every number for which more than half of its digits are digits 2, in increasing order:

$$2, 22, 122, 202, 212, 220, 221, 222, 223, 224, \dots$$

If the n th term in the list is 2022, what is n ?

Solution. Let's call the numbers for which more than half of their digits are 2 *nice* numbers. There is one one-digit nice number and one two-digit nice number. A three-digit nice number is either 222 or has two digits equal to 2 and one distinct digit, therefore there are

$$1 + 2 \times 9 + 8 = 27$$

three-digit nice numbers.

Now, a four-digit nice number is either 2222 or has exactly one distinct digit. So the initial segment of four-digit nice numbers is

$$1222, 2022, 2122, 2220, 2221, 2222, \dots$$

hence 2022 is the $1 + 1 + 27 + 2 = 31$ st term.

□

§3.3 Primary Problem 3

Prove that

$$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \dots + n(n+3) = \frac{n(n+1)(n+5)}{3}$$

for all positive integers n .

Solution 1. Writing in summation form, we have

$$\begin{aligned}
 & \sum_{k=1}^n k(k+3) \\
 &= \sum_{k=1}^n k^2 + 3k \\
 &= \sum_{k=1}^n k^2 + \sum_{k=1}^n 3k \\
 &= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} \\
 &= \frac{n(n+1)(2n+1+9)}{6} \\
 &= \frac{n(n+1)(n+5)}{3}
 \end{aligned}$$

as needed. □

Solution 2. We will induct on n . Clearly,

$$1 \cdot 4 = 4 = \frac{1(2)(6)}{3}.$$

Suppose $\sum_{k=1}^n k(k+3) = \frac{n(n+1)(n+5)}{3}$ holds for some $n \geq 1$, then

$$\begin{aligned}
 \sum_{k=1}^{n+1} k(k+3) &= \frac{n(n+1)(n+5)}{3} + (n+1)(n+4) \\
 &= \frac{(n+1)[n(n+5) + 3(n+4)]}{3} \\
 &= \frac{(n+1)[n^2 + 8n + 12]}{3} \\
 &= \frac{(n+1)(n+2)(n+6)}{3}
 \end{aligned}$$

and the induction is complete. □

Solution 3. Continuing from the second line of solution 1, notice that

$$k^2 + 3k = k(k+1) + 2k.$$

We handle each sum separately. For $\sum_{k=1}^n k(k+1)$, notice that the sum can be telescoped

$$\begin{aligned}
 \sum_{k=1}^n k(k+1) &= \sum_{k=1}^n \frac{k(k+1)((k+2) - (k-1))}{3} \\
 &= \sum_{k=1}^n \frac{(k)(k+1)(k+2)}{3} - \frac{(k-1)(k)(k+1)}{3} \\
 &= \frac{n(n+1)(n+2)}{3}
 \end{aligned}$$

whereas $\sum_{k=1}^n 2k = n(n+1)$ via Gaussian pairing. We may then add the sums, and the result follows. $\square$

Remark. We note that the entire sum may be telescoped straight away by noticing that

$$2k = 2 \left[\frac{k(k+1)}{2} - \frac{(k-1)(k)}{2} \right] = k(k+1) - (k-1)(k)$$

and

$$\begin{aligned} & \left[\frac{(k)(k+1)(k+2)}{3} - \frac{(k-1)(k)(k+1)}{3} \right] + [k(k+1) - (k-1)(k)] \\ &= \frac{k(k+1)(k+5)}{3} - \frac{(k-1)(k)(k+4)}{3} \end{aligned}$$

which telescopes nicely. Though we thought this may be a bit excessive as the sum of even numbers can be easily obtained via the Gaussian pairing trick instead.

§3.4 Primary Problem 4, Junior Problem 1

Given a circle and a quadrilateral $ABCD$ whose vertices all lie on the circle. Let R be the midpoint of arc AB . The line RC meets line AB at point S , and the line RD meets line AB at point T .

Prove that $CDTS$ is a cyclic quadrilateral.

Note: All vertices of a cyclic quadrilateral lie on a circle.

Solution. Note that

$$\begin{aligned} \angle STD &= \angle TDA + \angle DAT = \angle RDA + \angle DAB \\ &= \angle RCB + 180^\circ - \angle BCD = 180^\circ - \angle RCD = 180^\circ - \angle SCD \end{aligned}$$

and we're done. $\square$

§3.5 Primary Problem 5, Junior Problem 2

Without using a calculator, determine which number is greater: 17^{24} or 31^{19} .

Solution. Taking $\log_2$ on both sides and dividing by $19 \log_2 17$ yields $\frac{24}{19}$ and $\frac{\log_2 31}{\log_2 17}$, respectively. Then

$$\frac{\log_2 31}{\log_2 17} < \frac{\log_2 32}{\log_2 16} = \frac{5}{4} = \frac{25}{20} < \frac{24}{19}$$

thus $17^{24} > 31^{19}$. $\square$

Remark. The usual thing we do in question like this is to take logs, but picking a good base is crucial here. Noticing the proximity of 17 and 31 to powers of two makes the choice clear.

§3.6 Primary Problem 6, Junior Problem 3

Given an integer n . We rearrange the digits of n to get another number m .

Prove that it is impossible to get $m + n = 999999999$.

Solution. Note that n has 9 or fewer digits. Suppose that it is indeed possible, and $n = \overline{abcdefghi}$ and $m = \overline{opqrstuvw}$ where $o, p, q, r, s, t, u, v, w$ is a permutation of $a, b, c, d, e, f, g, h, i$. Then $i + w = 9$, otherwise the last digit of the sum cannot be 9. Similarly, $a + o = b + p = c + q = d + r = e + s = f + t = g + u = h + v = i + w = 9$. Hence

$$(a + b + c + d + e + f + g + h + i) + (o + p + q + r + s + t + u + v + w) = 81$$

but $(a + b + c + d + e + f + g + h + i) + (o + p + q + r + s + t + u + v + w) = 2(a + b + c + d + e + f + g + h + i)$ is even, a contradiction, and we're done. $\square$

§3.7 Junior Problem 4, Senior Problem 1

Given a circle Γ with center I , and a triangle ABC with all its sides tangent to Γ . A line ℓ is drawn such that it bisects both the area and the perimeter of triangle ABC .

Prove that line ℓ passes through I .

Solution. Suppose ℓ pass through points D and E on sides AB and AC of the triangle, relabel the vertices if necessary. Let r be the radius of Γ . Then

$$2|ADE| = |ABC|$$

and

$$2(AD + AE) = AB + BC + CA.$$

But

$$|ABC| = \frac{1}{2}(AB + BC + CA)(r)$$

hence

$$\begin{aligned} 2|ADE| &= (AD + AE)r \\ |ADE| &= \frac{r}{2}(AD + AE). \end{aligned}$$

Let P be the point on DE such that AP bisects $\angle BAC$. Note that the perpendicular distances from P to AD and AE are equal. Let that distance be x . Then

$$|ADE| = \frac{x}{2}(AD + AE)$$

hence $x = r$. But there is only one unique point on the angle bisector of $\angle BAC$ that has a perpendicular distance of r to the sides, which is I , hence $I = P$ lies on DE , i.e. on ℓ , and we're done. $\square$

§3.8 Junior Problem 5, Senior Problem 2

It is known that there are n integers $a_1, a_2, \dots, a_n$ such that

$$a_1 + a_2 + \dots + a_n = 0 \quad \text{and} \quad a_1 \times a_2 \times \dots \times a_n = n.$$

Determine all possible values of n .

Solution. We show that only integers n divisible by 4 are possible. First we construct the sequence for each multiple of 4. Let $n = 2^s M$, where M is odd and $s \geq 2$.

If $s = 2$, take

$$a_1 = a_2 = \dots = a_{2M-2} = -1, a_{2M-1} = 2M, a_{2M} = -2$$

and $a_k = 1$ if k is odd, and $a_k = -1$ if k is even for $k = 2M + 1, \dots, 4M$.

If $s = 3$, take

$$a_1 = \dots = a_{4M+2} = -1, a_{4M+3} = 4M, a_{4M+4} = 2.$$

After that, if $M > 1$ then take $a_k = 1$ if k is odd, and $a_k = -1$ if k is even for $k = 4M + 5, \dots, 8M$.

If $s \geq 4$, take

$$a_1 = \dots = a_{2^{s-2}M-4} = -1, a_{2^{s-2}M-3} = 2^{s-2}M, a_{2^{s-2}M-2} = -4$$

and $a_k = 1$ if k is odd, and $a_k = -1$ if k is even for $k = 2^{s-2}M - 1, \dots, 2^s M$.

For small M some of these ranges are empty, and an empty range is simply omitted. This happens for the block of -1 's when $s = 2$, $M = 1$ (since $2M - 2 = 0$) and when $s = 4$, $M = 1$ (since $2^{s-2}M - 4 = 0$), and for the alternating block when $s = 3$, $M = 1$. In all other cases the ranges are non-empty, since $2M - 2 \geq 4$ when $M \geq 3$ and $2^{s-2}M - 4 \geq 4$ when $s \geq 5$ or $M \geq 3$. Explicitly, the smallest cases are

$$\begin{aligned} n = 4: & \quad (a_1, \dots, a_4) = (2, -2, 1, -1), \\ n = 8: & \quad (a_1, \dots, a_8) = (-1, -1, -1, -1, -1, -1, 4, 2), \\ n = 16: & \quad (a_1, \dots, a_{16}) = (4, -4, 1, -1, 1, -1, \dots, 1, -1). \end{aligned}$$

One may easily check each construction works. Now we show that odd n or singly even n has no such sequence. For odd n , we see that each a_i is odd, thus the sum must be odd, i.e. cannot be equal to 0. For even n such that $4 \nmid n$, exactly one a_i is even, and the others odd. Since n is even, the sum must be odd as well, and we're done. $\square$

§3.9 Junior Problem 6, Senior Problem 3

Prove that there is a multiple of 2^{2022} that has 2022 digits, and can be written using digits 1 and 2 only.

Solution. We shall prove that for all positive integers n there exists a multiple of 2^n with exactly n digits, all of which are either 1 or 2. We shall induct on n . The case $n = 1$ is clear.

Suppose the statement holds for some $n \geq 1$. Let M be an n -digit number with only digit 1 and 2, such that $M = 2^n k$ for some integer k . Notice that for any integer a , we have

$$M + a10^n = 2^n(k + a5^n).$$

If k is odd, then we take $a = 1$, otherwise take $a = 2$. Either way we have $2 \mid k + a5^n$ and we're done. $\square$

§3.10 Senior Problem 4

Given a pentagon $ABCDE$ with all its interior angles less than 180° .

Prove that if $\angle ABC = \angle ADE$ and $\angle ADB = \angle AEC$, then $\angle BAC = \angle DAE$.

Solution. Let F be the intersection point of BD and CE . Then

$$\angle ADF = \angle ADB = \angle AEC = \angle AEF$$

thus $AFDE$ is cyclic. Moreover, from

$$\angle AFE = \angle ADE = \angle ABC$$

we see that $ABCF$ is also cyclic. Therefore

$$\angle BAC = \angle BFC = \angle DFE = \angle DAE$$

as needed.

§3.11 Senior Problem 5

Let a , b , r , and s be positive integers ($a \geq 2$), where a and b have no common prime factor.

Prove that if $a^r + b^r$ is divisible by $a^s + b^s$, then r is divisible by s .

Solution. By the division algorithm, take integer u and $v \in \{0, \dots, s-1\}$ such that $r = us + v$. Note that $a^s + b^s$ is coprime to a^s . Then

$$\begin{aligned} a^r &\equiv -b^r && (\text{mod } a^s + b^s) \\ \implies (a^s)^u (a^v) &\equiv -(b^s)^u (b^v) && (\text{mod } a^s + b^s) \\ \implies (a^s)^u (a^v) &\equiv -(-a^s)^u (b^v) && (\text{mod } a^s + b^s) \\ \implies a^v &\equiv (-1)^{u+1} b^v && (\text{mod } a^s + b^s) \end{aligned}$$

hence $a^s + b^s \mid a^v + b^v$ or $a^s + b^s \mid a^v - b^v$, either way $v \geq s$ unless $v = 0$, and we're done. $\square$

Remark. In fact, by looking closer at the last part of the proof, one may show that u is odd; otherwise we have $a^s + b^s \mid 2$ which is impossible since $a^s + b^s \geq 2 + 1 > 2$.

§3.12 Senior Problem 6

A football league has n teams. Each team plays one game with every other team. Each win is awarded 2 points, each tie 1 point, and each loss 0 point.

After the league is over, the following statement is true: for every non-empty subset S of teams in the league, there is a team (which may or may not be in S) such that the total points the team obtained by playing all the teams in S is odd.

Prove that n is even.

Solution. Since wins and losses award even number of points, they do not affect the parity. Moreover, ties are mutual between teams. Therefore, consider a graph $G = (V, E)$ on the set of teams V such that $ab \in E$ if and only if team a and b tied. The condition now translates to: for all nonempty $S \subset V$, there exists $v \in V$ such that $|N(v) \cap S|$ is odd, where $N(v)$ is the set of neighbours of v .

Let f be a function on the subsets of V such that $f(S)$ is the set of all vertices that have an odd number of neighbours in S . Consider two subsets $S, T \subset V$. If $u \in f(S) \cap f(T)$, then $|N(u) \cap (S \setminus T)| \equiv |N(u) \cap (T \setminus S)| \pmod{2}$, thus $u \notin f(S \Delta T)$, where $S \Delta T$ is the symmetric difference of S and T . Therefore

$$f(S \Delta T) \subset f(S) \Delta f(T).$$

Therefore f is injective, otherwise we have two distinct subsets S and T where $f(S) = f(T)$, and

$$f(S \Delta T) \subset f(S) \Delta f(T) = \emptyset$$

which is impossible since $S \Delta T \neq \emptyset$. Moreover, since the family of subsets of V is finite, f is also surjective, thus also bijective.

Now, for each $v \in V$, consider $f^{-1}(\{v\})$, which is well-defined. Suppose $u \in f^{-1}(\{v\})$, and consider $f^{-1}(\{u\})$. Since every vertex in $f^{-1}(\{v\})$ has an even number of neighbours in $f^{-1}(\{u\})$, except u itself which has odd number of neighbours, the number of edges between $f^{-1}(\{u\})$ and $f^{-1}(\{v\})$ must be odd. Therefore we must have $v \in f^{-1}(\{u\})$ otherwise every vertex in $f^{-1}(\{u\})$ has even number of neighbours in $f^{-1}(\{v\})$, which gives an even number of edges, a contradiction. So we have

$$u \in f^{-1}(\{v\}) \iff v \in f^{-1}(\{u\}).$$

Finally, consider P the number of ordered pairs of vertices (u, v) such that $uv \in E$ and $u \in f^{-1}(\{v\})$. Clearly, P is even since if (u, v) is counted then (v, u) is counted. Now, for each vertex v , the number of vertices u such that (u, v) is counted is the number of neighbours of v in $f^{-1}(\{v\})$, which is odd. Therefore the parity of P is the same as the parity of $n = |V|$, which must be even, and we're done. $\square$

§4 2023

§4.1 Primary Problem 1

Prove that for all positive integers n , $1^3 + 2^3 + 3^3 + \cdots + n^3$ is a perfect square.

Solution. We show by induction that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$ and therefore is a perfect square.

Base case:

When $n = 1$, $LHS = RHS = 1$.

Inductive step:

Given that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2} \right]^2$$

Then

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2} \right]^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2 \left[\frac{k^2}{4} + k + 1 \right] \\ &= (k+1)^2 \left[\frac{k^2 + 4k + 4}{4} \right] \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \left[\frac{(k+1)(k+2)}{2} \right]^2 \end{aligned}$$

End of induction.

§4.2 Primary Problem 2

Ivan bought 50 cats such that each cat can be categorized into exactly one of five different colors. He records the number of cats of each color and after multiplying these five numbers he obtains the number 100000. How many cats of each color does he have?

Solution. By AM-GM inequality, if a_i denote the number of cats of each color, then $a_1 + a_2 + a_3 + a_4 + a_5 = 50$ implies

$$\begin{aligned} \sqrt[5]{a_1 a_2 a_3 a_4 a_5} &\leq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} = 10 \\ \Rightarrow a_1 a_2 a_3 a_4 a_5 &\leq 100000 \end{aligned}$$

But since equality holds in the problem statement, then we must have the equality case for AM-GM, that is $a_1 = a_2 = \cdots = a_5 = 10$ must hold.

§4.3 Primary Problem 3

Find all pairs of positive integers (x, y) , so that the number $x^3 + y^3$ is a prime.

Solution. Factorizing $x^3 + y^3 = (x + y)(x^2 + y^2 - xy)$, we see that $x + y \geq 2$. So if $x^3 + y^3$ is a prime, then $x^2 + y^2 - xy = 1$ must hold. Then $(x - y)^2 + xy = 1$, but $xy \geq 1$, so $(x - y)^2 = 0$ and $xy = 1$ must hold, ie $x = y = 1$ is the only solution.

§4.4 Primary Problem 4

Given a right-angled triangle with hypotenuse 2024, find the maximal area of the triangle.

Answer. 1012^2

Let a and b be the two other side lengths of the right-angled triangle.

$$2024^2 = a^2 + b^2 \geq 2ab$$

$$Area = \frac{ab}{2} \leq 1012^2$$

Equality case: $a = b = 1012\sqrt{2}$

§4.5 Primary Problem 5

Ruby writes the numbers $1, 2, 3, \dots, 10$ on the whiteboard. In each move, she selects two distinct numbers, a and b , erases them, and replaces them with $a + b - 1$. She repeats this process until only one number, x , remains. What are all the possible values of x ?

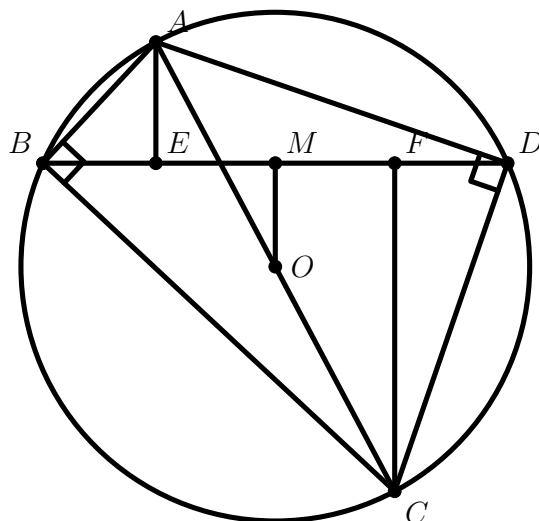
Answer. 46

Since there are initially 10 numbers, Ruby will perform exactly 9 moves until the single x remains. However, note that the total sum of the numbers on the whiteboard decreases by 1 after each move. Since the initial sum is $1 + 2 + 3 + \dots + 10 = 55$, the final sum, which is x , will be $55 - 9 = 46$.

§4.6 Primary Problem 6

Suppose $ABCD$ is a cyclic quadrilateral with $\angle ABC = \angle ADC = 90^\circ$. Let E and F be the feet of perpendiculars from A and C to BD respectively. Prove that $BE = DF$.

Solution.



Let O, M be the center of $(ABCD)$ and the midpoint of BD respectively. $OM \perp BD$ since $OB = OD$. $AE \parallel OM \parallel FC$ and $OA = OC$ implies $ME = MF$. Therefore $BE = MB - ME = MD - MF = DF$.

§4.7 Senior Problem 1

Prove that there exists infinitely many positive integers that can be written as the sum of 2001 distinct perfect squares and as the sum of 2023 distinct positive perfect cubes.

Solution. Note that $(1 + 2 + 3 + \cdots + 23)^2 = 1^3 + 2^3 + 3^3 + \cdots + 23^3$. Adding 2000 sixth powers to both sides,

$$(1 + 2 + 3 + \cdots + 23)^2 + a_1^6 + a_2^6 + a_3^6 + \cdots + a_{2000}^6 = 1^3 + 2^3 + 3^3 + \cdots + 23^3 + a_1^6 + a_2^6 + a_3^6 + \cdots + a_{2000}^6$$

where $1 + 2 + 3 + \cdots + 23 < a_1 < a_2 < a_3 < \cdots < a_{2000}$ (to ensure all the perfect squares/cubes are distinct). We observe that the left hand side is the sum of 2001 perfect squares, while the right hand side is the sum of 2023 perfect cubes. We can change the a_i s freely to obtain infinitely many such numbers.

§4.8 Senior Problem 2

Prove that if x, y, n are positive integers such that

$$x^{2024} + y^{2024} = 2^n$$

then $x = y$.

Solution. (Kwong Weng)

If $n = 1$, then obviously $x = y = 1$. Now we assume that $n > 1$. Note that x, y must be either both even or both odd. If both x, y are even, then we can divide each of x, y by 2 and replace n by $n - 2024$. (Essentially we are dividing both sides by 2^{2024}). We can

repeat this until x is odd, in which y must be odd too. Since perfect squares is congruent to 0 or 1 modulo 4, we have that

$$2^n \equiv x^{2024} + y^{2024} \equiv 1 + 1 \equiv 2 \pmod{4}$$

Therefore $n = 1$, so $x = y = 1$, and we have $x = y$ initially.

§4.9 Senior Problem 3

Fix an integer $n \geq 1$. On a $m \times n$ chessboard, what is the minimum value of m such that n queens can be placed on the chessboard without any two attacking each other? (A queen can attack vertically, horizontally, or diagonally across any distance.)

Solution. $m = n$ for $n = 1$ and $n \geq 4$, $m = n + 1$ for $n = 2, 3$.

Solve cases $n = 1, 2, 3$ manually.

For $n = \text{even} \geq 4$, let $n = 2k$.

- When $k \equiv 0$ or $2 \pmod{3}$, place the queens in squares $(1, 2), (2, 4), (3, 6) \dots (k, 2k)$ and $(k + 1, 1), (k + 2, 3) \dots (2k, 2k - 1)$.
- When $k \equiv 1 \pmod{3}$, place the queens in squares $(1, 2), (2, 4), (3, 6) \dots (k, 2k)$ and $(k + 1, 3), (k + 2, 1)$ and $(k + 3, 7), (k + 4, 9) \dots (2k - 1, 2k - 1)$ and $(2k, 5)$.

For $n = \text{odd} \geq 5$, if $n \equiv 1$ or $5 \pmod{6}$, let $n = 2k + 1$. Add an extra row 0 and a column 0 to the $n = 2k$ configuration, then place a queen at $(0, 0)$. Note that these configurations work as for every queen (x, y) in a configuration, the values of $x, y, x + y$ and $x - y$ are all unique.

If $n \equiv 3 \pmod{6}$, then place the queens in $(1, 4), (2, 6), \dots, (k - 1, 2k), (k, 2), (k + 1, 5), (k + 2, 7), \dots, (2k - 1, 2k + 1), (2k, 1), (2k + 1, 3)$. Note that these configurations work as for every queen (x, y) in a configuration, the values of $x, y, x + y$ and $x - y$ are all unique.

§4.10 Senior Problem 4

Given a right angled triangle ABC with $\angle BAC = 90^\circ$. The points D, E, F lie on sides BC, CA, AB respectively so that AD is perpendicular to BC and EF is parallel to BC . A point G lies on side AC such that $AG = CE$.

Help Ivan prove that $\angle GDF = 90^\circ$.

Solution. (Kwong Weng)

Note that

$$\frac{AG}{BF} = \frac{CE}{BF} = \frac{CA}{BA} = \tan \angle ABC = \frac{AD}{BD}$$

This implies that $\frac{AG}{AD} = \frac{BF}{BD}$. Combining this result with $\angle DAG = 90^\circ - \angle ACB = \angle DBF$, we have that $\triangle AGD \sim \triangle BFD$. Therefore

$$\angle AGD = \angle BFD = 180^\circ - \angle AFD$$

which means A, F, D, G are cyclic and therefore $\angle GDF = 180^\circ - \angle GAF = 90^\circ$.

§4.11 Senior Problem 5

Find the smallest positive real m such that if a, b, c are three side lengths of a triangle with $a^2 + b^2 > mc^2$, then c must be the shortest side.

Solution. (Kwong Weng)

Answer. 5

WLOG suppose $x \geq y \geq z$ be three side lengths of a triangle. By triangle inequality $2y \geq x$, so

$$x^2 + z^2 \leq (2y)^2 + y^2 = 5y^2 \leq 5x^2$$

This means that if $a^2 + b^2 > 5c^2$, then c must be the shortest side.

Now we show that $m < 5$ fails. In other words, we show that there exist $x \geq y \geq z$ which form three side lengths of the triangle such that

$$x^2 + z^2 > my^2$$

Take

$$y = 1, \quad z = 1 - \varepsilon, \quad x = 2 - 2\varepsilon$$

for sufficiently small $\varepsilon > 0$. Then $x > y > z$, so z is the shortest side, and these form a triangle since $y + z = 2 - \varepsilon > x$. Moreover,

$$x^2 + z^2 = (2 - 2\varepsilon)^2 + (1 - \varepsilon)^2 = 5(1 - \varepsilon)^2,$$

which is greater than $m = my^2$ whenever $\varepsilon < 1 - \sqrt{m/5}$. This is possible since $m < 5$. Taking $(a, b, c) = (x, z, y)$ gives $a^2 + b^2 > mc^2$ while c is not the shortest side.

§4.12 Senior Problem 6

Ivan has a parallelogram whose interior angles are $60^\circ, 120^\circ, 60^\circ, 120^\circ$ respectively, and all side lengths are integers. Is it possible that one of the diagonals has length $\sqrt{2024}$?

Solution. (Kwong Weng)

Suppose the two side lengths of the parallelogram are x and y . By Cosine Rule, the two diagonals have lengths $\sqrt{x^2 + xy + y^2}$ and $\sqrt{x^2 - xy + y^2}$ respectively. We will only consider if there exists integers x, y such that $x^2 + xy + y^2 = 2024$ as the other form is equivalent to replacing y with $-y$. Suppose 3 divides x . Then

$$2 \equiv 2024 \equiv x^2 + xy + y^2 \equiv y^2 \pmod{3}$$

which is impossible since perfect squares are only equivalent to 0, 1 modulo 3. By symmetry 3 cannot divide y . Therefore

$$2 \equiv x^2 + xy + y^2 \equiv 1 + xy + 1 \pmod{3} \implies xy \equiv 0 \pmod{3}$$

which is a contradiction as we assumed that x, y are not divisible by 3.

§5 2024

§5.1 Primary Problem 1

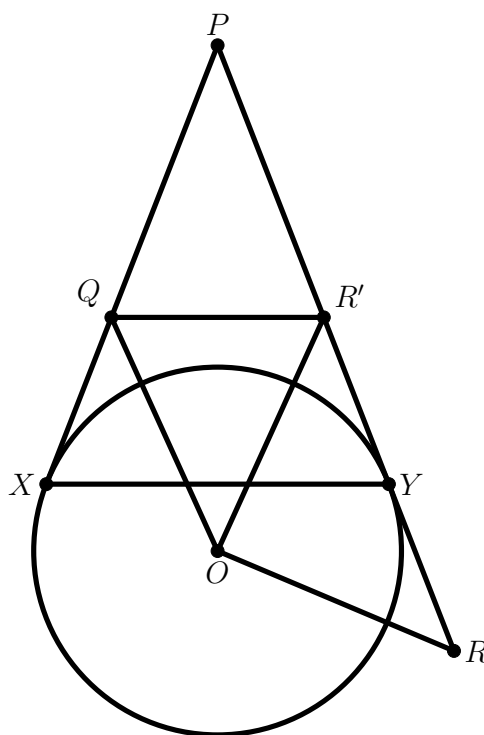
A string of letters is called good if it contains a continuous substring *IMONST* in it. For example, the string *NSIMONSTIM* is good, but the string *IMONNNST* is not. Find the number of good strings consisting of 12 letters from *I, M, O, N, S, T* only.

Solution. The continuous substring *IMONST* can appear in $12 - 6 + 1 = 7$ different positions in a string with 12 letters. Since the other letters can be chosen freely, the number of good strings of length 12 is $7 \cdot 6^6 - 1$. This is because *IMONSTIMONST* is the only good string that is counted twice, and all other strings can only contain at most one substring *IMONST* in it.

§5.2 Primary Problem 2

Jia Herng has a circle ω with center O , and P is a point outside of ω . Let PX and PY are two lines tangent to ω at X and Y , and Q is a point on segment PX . Let R is a point on the ray PY beyond Y such that $QX = RY$.

Help Jia Herng prove that the points O, P, Q, R are concyclic.



Solution 1. Let R' be R reflected across Y . QR' is parallel to XY due to R' and Y being reflections of Q and X across line OP respectively. We have $\angle ORP = \angle ORY = \angle OR'Y = 180^\circ - \angle OR'P = 180^\circ - \angle OQP$. So, $OPQR$ is cyclic.

Solution 2. Note that triangles OXQ and OYR are congruent, because $OX = OY$, $QX = RY$, and $\angle OXQ = \angle OYR = 90^\circ$. Subsequently, $\angle OQX = \angle YRO$, which implies $OPQR$ is cyclic.

§5.3 Primary Problem 3

Janson wants to find a sequence of positive integers $a_1, a_2, \dots, a_{2024}$ such that each term is at least 10, and a_i has exactly a_{i+1} divisors for all $1 \leq i \leq 2023$.

Can you help him find one such sequence, or is this task impossible?

Solution. We claim, the answer is yes. Set $a_{2024} = 10$, and $a_i = 2^{a_{i+1}-1}$ for all $i = 2023, 2022, \dots, 1$. Since 2 is a prime, a_i has $(a_{i+1} - 1) + 1$ factors as desired.

§5.4 Primary Problem 4, Junior Problem 1

Pingu is given two positive integers m and n without any common factors greater than 1.

- a) Help Pingu find positive integers p, q such that

$$\gcd(pm + q, n) \cdot \gcd(m, pn + q) = mn$$

- b) Prove to Pingu that he can never find positive integers r, s such that

$$\text{lcm}(rm + s, n) \cdot \text{lcm}(m, rn + s) = mn$$

regardless of the choice of m and n .

Solution. For part a), taking $p = q = mn$ gives $\gcd(pm + q, n) = n$ and $\gcd(m, pn + q) = m$. Hence their product is mn .

For part b), we note that $\text{lcm}(a, b) \geq \max\{a, b\}$ for any positive integers a, b with equality if and only if one divides the other. Suppose such r, s exists, then we must have $\text{lcm}(rm + s, n) \geq n$ and $\text{lcm}(m, rn + s) \geq m$. This means that equalities holds in both inequalities, implying $rm + s \mid n$ and $rn + s \mid m$. We now have the bound

$$m + n \geq (rm + s) + (rn + s) = r(m + n) + 2s \geq m + n + 2$$

since $r, s \geq 1$, a contradiction.

§5.5 Primary Problem 5, Junior Problem 2

A Duck drew a square $ABCD$, then he reflected C across B to obtain a point E . He also drew the center of the square to be F . Then, he drew a point G on ray EF beyond F such that $\angle AGC = 135^\circ$.

Help the Duck prove that $\angle CGD = 135^\circ$ as well.

Solution.

would have already jumped at least a full circle, while the red frog haven't yet. This is a contradiction.

Similarly, if the blue frog only passed by the red frog once, then when the red frog completes a full revolution, the blue frog have not completed a full circle yet. This is a contradiction.

So the frogs will pass by each other exactly twice. In both occasions mark a point where both frogs crossed while passing each other, then draw the line through these two points will suffice.

§5.7 Junior Problem 4, Senior Problem 1

Suppose a, b, c, d are positive reals such that $a \geq b \geq c \geq d$ and $ab^2c^3d^4 = 1$. Help Janson prove that $a + b + c + d \geq 4$.

Solution. By AM-GM,

$$\begin{aligned} \frac{a + 2b + 3c + 4d}{10} &= \frac{a + b + b + c + c + c + d + d + d + d}{10} \geq (ab^2c^3d^4)^{1/10} = 1 \\ &\Rightarrow \frac{a + 2b + 3c + 4d}{5} + \frac{a + 2b + 3c + 4d}{5} \geq 4 \end{aligned}$$

By rearrangement inequality,

$$\begin{aligned} \frac{a + 2b + 3c + 4d}{5} + \frac{4a + 3b + 2c + d}{5} &\geq \frac{a + 2b + 3c + 4d}{5} + \frac{a + 2b + 3c + 4d}{5} \geq 4 \\ &\Rightarrow \frac{5a + 5b + 5c + 5d}{5} = a + b + c + d \geq 4. \end{aligned}$$

Comment. Alternatively, one can notice that $4a + 3b + 2c + d \geq a + 2b + 3c + 4d \iff 3(a - d) + (b - c) \geq 0$, which is true as $a \geq d$ and $b \geq c$.

§5.8 Junior Problem 5, Senior Problem 2

A sequence of integers $a_1, a_2, \dots$ is called good if:

- $a_1 = 1$, and;
- $a_{i+1} - a_i$ is either 1 or 2 for all $i \geq 1$.

Find all positive integers n that cannot be written as a sum $n = a_1 + a_2 + \dots + a_k$, such that the integers $a_1, a_2, \dots, a_k$ forms a good sequence.

Solution. Only 2 and 5 are the answers.

- 2 cannot be written as such because $a_1 = 1 < 2$ and $a_1 + a_2 \geq 1 + 2 > 2$.
- 5 cannot be written as such because $a_1 = 1 < 5$, $a_1 + a_2 \leq 1 + 3 < 5$ and $a_1 + a_2 + a_3 \geq 1 + 2 + 3 > 5$.

If $n = 1, 3$ or 4 , then $1 = 1$, $3 = 1 + 2$ and $4 = 1 + 3$. Now we claim that all $n \geq 6$ can be written as $a_1 + a_2 + \cdots + a_k$. Let k be the largest positive integer satisfying $1 + 2 + \cdots + k \leq n$. Then since $n \geq 6 = 1 + 2 + 3$, then $k \geq 3$. Write $n = (1 + 2 + \cdots + k) + i$. Then $i \leq k$, otherwise $n \geq 1 + 2 + \cdots + (k + 1)$, contradicting the maximality of k .

Case 1: $i = 0$

Then $n = 1 + 2 + \cdots + k$. So we can take the sequence $a_i = i$ for all $1 \leq i \leq k$.

Case 2: $0 < i < k$

We can take $a_1 = 1$, $a_2 = 2$, ..., $a_{k-i} = k - i$, followed by $a_{k-i+1} = k - i + 2$, $a_{k-i+2} = k - i + 3$, ..., $a_k = k + 1$. The sum is $(1 + 2 + \cdots + k) + i = n$.

Case 3: $i = k$

Then $n = (1 + 2 + 3 + \cdots + k) + k = 1 + 3 + 4 + \cdots + k + (k + 2)$, which works for all $k \geq 3$. So we can take the sequence $a_1 = 1$, $a_i = i + 1$ for all $2 \leq i \leq k - 1$, and $a_k = k + 2$.

§5.9 Junior Problem 6, Senior Problem 3

Ivan claims that for all positive integers n ,

$$\left\lfloor \sqrt{\frac{n}{1^3}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{2^3}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{3^3}} \right\rfloor + \cdots = \left\lfloor \sqrt[3]{\frac{n}{1^2}} \right\rfloor + \left\lfloor \sqrt[3]{\frac{n}{2^2}} \right\rfloor + \left\lfloor \sqrt[3]{\frac{n}{3^2}} \right\rfloor + \cdots$$

Why is he correct? (Here, $\lfloor x \rfloor$ denotes the floor function.)

Solution 1. Each $\left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor$ counts the number of positive integers j such that $j \leq \sqrt{\frac{n}{i^3}}$ or $i^3 j^2 \leq n$, while each $\left\lfloor \sqrt[3]{\frac{n}{i^2}} \right\rfloor$ counts the number of positive integers j such that $j \leq \sqrt[3]{\frac{n}{i^2}}$ or $i^2 j^3 \leq n$, but these are both counting the number of ordered pairs of positive integers (i, j) such that $i^2 j^3 \leq n$. Hence, LHS = RHS.

Solution 2. Let

$$f(n) = \left\lfloor \sqrt{\frac{n}{1^3}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{2^3}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{3^3}} \right\rfloor + \cdots$$

and

$$g(n) = \left\lfloor \sqrt[3]{\frac{n}{1^2}} \right\rfloor + \left\lfloor \sqrt[3]{\frac{n}{2^2}} \right\rfloor + \left\lfloor \sqrt[3]{\frac{n}{3^2}} \right\rfloor + \cdots$$

We proceed by induction.

We have $f(1) = 1$ and $g(1) = 1$. Now, suppose that $f(n) = g(n)$ for some positive integer n . Notice that

$$\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor - \left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor = \begin{cases} 1 & \text{if } \sqrt{\frac{n+1}{i^3}} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

for all positive integers i . This is because $\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor \geq \left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor$, while if $\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor -$

$\left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor \geq 2$, then there exist a integer k such that

$$\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor < k < \left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor \implies \sqrt{\frac{n}{i^3}} < k < \sqrt{\frac{n+1}{i^3}} \implies n < i^3 k^2 < n+1$$

which is absurd. Therefore,

$$\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor - \left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor \in \{0, 1\}$$

and if $\left\lfloor \sqrt{\frac{n+1}{i^3}} \right\rfloor - \left\lfloor \sqrt{\frac{n}{i^3}} \right\rfloor = 1$ and $\sqrt{\frac{n+1}{i^3}}$ isn't an integer, then there must again exist an integer k such that

$$\sqrt{\frac{n}{i^3}} < k < \sqrt{\frac{n+1}{i^3}} \implies n < i^3 k^2 < n+1$$

which is absurd. Hence,

$$f(n+1) - f(n) = \text{number of positive integers } i \text{ such that } \frac{n+1}{i^3} \text{ is a perfect square}$$

Similarly,

$$g(n+1) - g(n) = \text{number of positive integers } i \text{ such that } \frac{n+1}{i^2} \text{ is a perfect cube}$$

This means that

$$f(n+1) - f(n) = \text{number of ways of expressing } n+1 \text{ as } i^2 j^3 \ (i, j \in \mathbb{N}) = g(n+1) - g(n)$$

Hence

$$f(n+1) = f(n) + (f(n+1) - f(n)) = g(n) + (g(n+1) - g(n)) = g(n+1)$$

and the induction is complete.

§5.10 Senior Problem 4

For all $n \geq 1$, define a_n to be the fraction $\frac{k}{2^n}$ such that a_n is the closest to $\frac{1}{3}$ over all integer values of k .

Prove that the sequence $a_1, a_2, \dots$ satisfies the equation $2a_{i+2} = a_{i+1} + a_i$ for all $i \geq 1$.

Solution. The problem is equivalent to finding k such that $\frac{3k}{2^n}$ is closest to 1. So we want the closest multiple of 3, $3k$, that is closest to 2^n . Then $3k$ can either be 2^n , $2^n - 1$ or $2^n + 1$, as one of these three integers must be a multiple of 3.

Now, note that $2^n \equiv_3 (-1)^n$, so the closest multiple of 3 to 2^n is $2^n - (-1)^n$. Thus, the fraction we seek is $a_n = \frac{2^n - (-1)^n}{3 \cdot 2^n}$. Now we have

$$\begin{aligned} a_{i+1} + a_i &= \frac{2^{i+1} - (-1)^{i+1}}{3 \cdot 2^{i+1}} + \frac{2^i - (-1)^i}{3 \cdot 2^i} \\ &= \frac{2^{i+2} + 2 \cdot (-1)^{i+2} + 2^{i+2} - 4 \cdot (-1)^{i+2}}{3 \cdot 2^{i+2}} \\ &= 2 \cdot \frac{2^{i+2} - (-1)^{i+2}}{3 \cdot 2^{i+2}} \\ &= 2a_{i+2}, \end{aligned}$$

which completes the proof.

§5.11 Senior Problem 5

Janson found 2025 dogs on a circle. Janson wants to select some (possibly none) of the dogs to take home, such that no two selected dogs have exactly two dogs (whether selected or not) in between them. Let S_1 be the number of ways for him to do so.

Ivan also found 2025 cats on a circle. Ivan wants to select some (possibly none) of the cats to take home, such that no two selected cats have exactly five cats (whether selected or not) in between them. Let S_2 be the number of ways for him to do so.

- Prove that $S_1 = S_2$.
- Prove that S_1 and S_2 are both perfect cubes.

Solution. a) Denote the dogs as $d_1, d_2, \dots, d_{2025}$, and denote the cats $c_1, c_2, \dots, c_{2025}$. We will establish a bijection where a dog d_n is picked by Janson if and only if the cat c_{2n} is picked by Ivan, where indices are taken modulo 2025.

Because 2025 is coprime to 2, then for any cat c_n Ivan picked, this happens if and only if the dog d_{an} is picked by Janson, where $a = 1013$ is the unique residue mod 2025 such that $2a \equiv 1 \pmod{2025}$.

Now, observe no two dogs fight in Janson's choice, then no two dogs d_i, d_j satisfies $|i - j| = 3$, which is $|2i - 2j| = 6$, hence the two cats c_{2i} and c_{2j} would not fight. Hence none of Ivan's cats will fight as well.

Finally, observe the converse holds as if no two cats c_i and c_j in Ivan's choice would fight, then $|i - j| = 6$ never hold, thus $|ai - aj| \equiv 3 \pmod{2025}$, so the two dogs d_{ai} and d_{aj} will not fight. Hence none of Janson's dogs will fight as well.

b) It suffices to show that S_1 is a perfect cube. Notice that if two dogs fight, their indices must be congruent mod 3. Therefore, we can separate S_1 into a product of the number of ways to select dogs for each residue mod 3 because dogs with indices of different residues mod 3 will not fight.

Moreover, for each residue class mod 3, the only dogs that will fight are dogs that differ by a distance of 3, and this condition is the same for each residue class with equal number

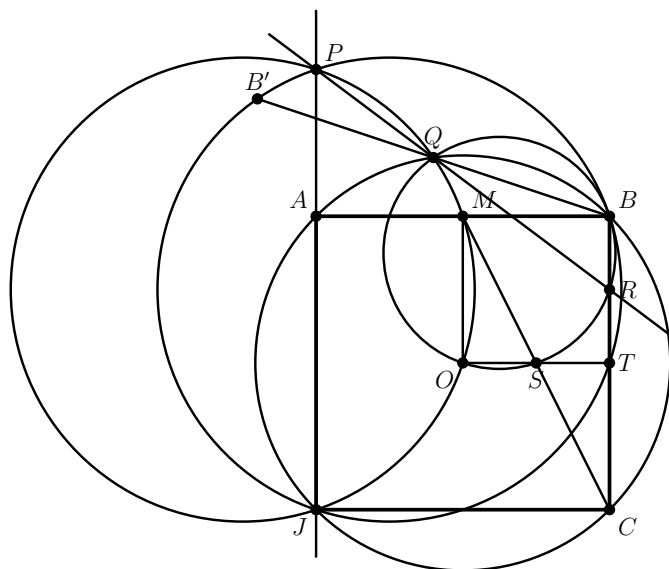
of dogs, as 2025 is divisible by 3. Since for each residue mod 3, there are the same number of ways to select the dogs, then if the number of ways to select dogs for each residue mod 3 is T , then $S_1 = T^3$ which is a perfect cube.

§5.12 Senior Problem 6

Rui Xuen has a circle ω with center O , and a square $AB CJ$ with vertices on ω . Let M be the midpoint of AB , and let Γ be the circle passing through the points J, O, M . Suppose Γ intersect line AJ at a point $P \neq J$, and suppose Γ intersect ω at a point $Q \neq J$. A point R lies on side BC so that $RC = 3RB$.

Help Rui Xuen prove that the points P, Q, R are collinear.

Common remarks. We redefine R as the intersection of lines PQ and BC , and we aim to prove that $RC = 3BR$. Let T be the midpoint of BC , thus showing R is the midpoint of BT suffices.



Claim. The points Q, M, C lie on the same line.

Proof. We intersect CM with ω again at a point Q' , and we show that $Q' \in \Gamma$. In fact, this is due to $\angle Q'MO = \angle CMO = \angle MCB = \angle Q'CB = \angle Q'JB = \angle Q'JO$. $\square$

Claim. The points Q, B, R, O lie on a circle γ .

Proof. $\angle OQR = \angle OQP = \angle OJP = 45^\circ = \angle OBR$. $\square$

Solution 1. Now, let S be the midpoint of OT which also lies on MC . I claim that S also lies on γ . This is because $\angle QSO = \angle QCJ = \angle QBJ = \angle QBO$. Finally, we see that $MBTO$ forms a square, implying $TR \cdot TB = TS \cdot TO = TS \cdot TB$, so $TR = TS = OT/2 = BT/2 \implies BR = RT$, completing the proof.

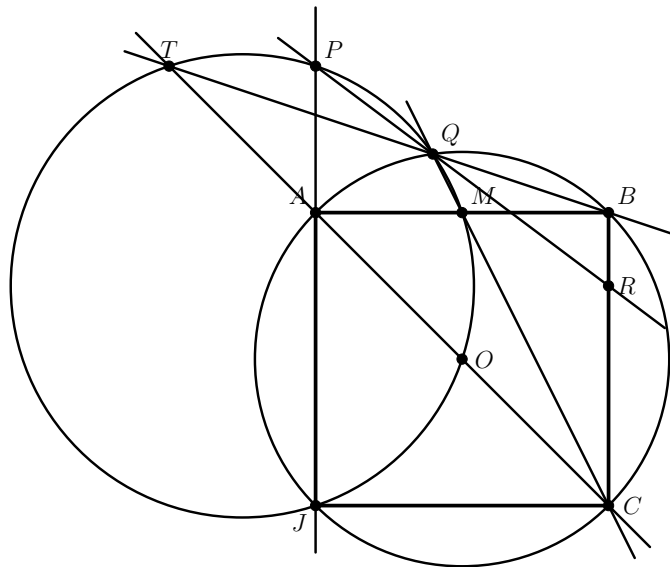
Solution 2. Consider the reflection B' of B about Q , so that $QO \parallel B'J$. We now have

$$\angle JB'B = \angle OQB = \angle QBO = \angle QBJ = \angle QCJ = \angle MCJ = \angle JTC = \angle JTB,$$

so $JTB'B'$ is inscribed in a circle Ω . Notice that the homothety (enlargement) centered at B with scale factor 2 sends Q to B' and O to J , implying that this homothety sends γ to Ω . Since the line BRT transverses γ and Ω at R and T respectively, the aforementioned homothety sends R to T , as desired.

Comment. It is easy to prove that P lies on Ω as well, by noticing that the quadrilaterals $JOMP$ and $BTJP$ forms isosceles trapezoids.

Solution 3.



Let the lines BQ and AC intersect at T . We proceed with the following claim.

Claim. T lies on Γ .

Proof. We angle chase that

$$\begin{aligned} \angle OTQ &= \angle OQB - \angle QOA = \angle QBO - 2\widehat{QA} = \widehat{QJ} - 2\widehat{QA} \\ &= \widehat{AJ} - \widehat{QA} = \widehat{AB} - \widehat{QA} = \widehat{QB} = \angle BJQ = \angle OJQ. \end{aligned} \quad \square$$

Let s be the side length of the square. Firstly we must have $CM = \frac{s\sqrt{5}}{2}$ and so,

$$MQ = \frac{AM \cdot MB}{CM} = \frac{\frac{s^2}{4}}{\frac{s\sqrt{5}}{2}} = \frac{s}{2\sqrt{5}},$$

meaning that

$$CQ = s \left(\frac{\sqrt{5}}{2} + \frac{1}{2\sqrt{5}} \right) = \frac{3s}{\sqrt{5}}.$$

We compute the ratio $\frac{BR}{CR}$ using the ratio lemma on $\triangle BQC$ and R , as well as the law of sines, namely

$$\begin{aligned}
 \frac{BR}{CR} &= \frac{BQ}{CQ} \cdot \frac{\sin \angle BQR}{\sin \angle RQC} \\
 &= \frac{\sin \angle BCM}{\sin \angle QJC} \cdot \frac{\sin \angle PQT}{\sin \angle PQM} \\
 &= \frac{\sqrt{2}s \cdot \sin \angle BCM}{QC} \cdot \frac{\sin \angle POA}{\sin \angle AJM} && \text{(Law of sines on } \triangle QJC) \\
 &= \frac{\sqrt{2}s \sin \angle POA}{\frac{3s}{\sqrt{5}}} \\
 &= \frac{\sqrt{10}}{3} \cdot \frac{PA \sin \angle JPO}{AO} && \text{(Sine rule on } \triangle POA) \\
 &= \frac{\sqrt{10}}{3} \cdot \frac{\sin \angle PJM}{\sqrt{2}} \\
 &= \frac{\sqrt{5}}{3} \cdot \frac{1}{\sqrt{5}} \\
 &= \frac{1}{3},
 \end{aligned}$$

completing the proof.

Comment. In fact, T also lies on line RM , and $TBRJ$ is a cyclic quadrilateral. Moreover, the tangent lines to Γ through T and O , as well as the lines MJ and BC all meet at a point.

§6 2025

§6.1 Beginner Problem 1

You Seng has numbers 0 to 101, and he forms 51 pairs of numbers in some way. Prove that no matter how he pairs the numbers, the number of pairs that has an odd number and an even number, is odd.

Solution 1. Let the number of even-even pairs and even-odd pairs be m and n respectively. Counting even number that each pair contribute, $2m + n = 51 \implies n \equiv 1 \pmod{2}$.

Solution 2. Let the number of odd-odd pair and even-odd pair be m and n respectively. Counting odd number that each pair contribute, $2m + n = 51 \implies n \equiv 1 \pmod{2}$.

§6.2 Beginner Problem 2

Do there exist 10 distinct positive integers $x_1, x_2, \dots, x_{10}$ such that if $S = x_1 + \dots + x_{10}$ is the sum of all of these numbers, then each of the 10 integers is a divisor of S ?

Answer. Yes.

Solution. Take

$$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 6, x_5 = 12, x_6 = 24, x_7 = 48, x_8 = 96, x_9 = 192, x_{10} = 384,$$

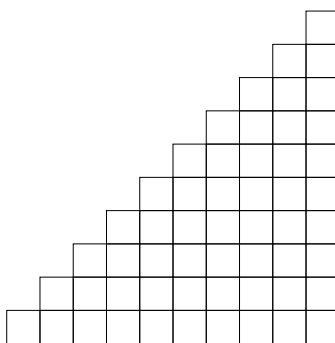
which works since

$$S = 1 + 2 + 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 = 768,$$

which is divisible by 1, 2, 3, 6, 12, 24, 48, 96, 192 and 384.

§6.3 Beginner Problem 3

Given a staircase with height 10, you have one tile of dimensions $1 \times 1, 1 \times 2, \dots, 1 \times 10$ each. How many ways can you form the staircase using these tiles?



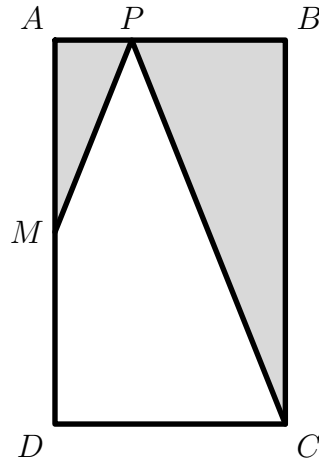
Answer. $2^9 = 512$ ways.

Solution. Notice that we have only 2 places to put the 1×10 tile, both of which result in the remaining space being a staircase with height 9. We can repeat this for $1 \times 9, 1 \times 8, \dots, 1 \times 2$, with there being 2 places to put the $1 \times k$ tile, both of which result in the remaining space being a staircase with height $k - 1$. We get $2^9 = 512$ ways.

§6.4 Beginner Problem 4

Let $ABCD$ be a rectangle with the length of $AB = 111$ cm. M is the midpoint of AD , and point P lies on AB such that the angles $\angle APM = \angle BPC$. Find the length of AP .

Solution. Because of the angle condition, triangles $\triangle APM$ and $\triangle BPC$ are similar.



Therefore,

$$\frac{AP}{PB} = \frac{AM}{BC} = \frac{1}{2}$$

so $AP = \frac{1}{3}AB = 37$ cm.

§6.5 Beginner Problem 5

Darien initially had two positive real numbers $a_0 > b_0$ on his whiteboard. He erased both numbers and replaced them with a new pair defined by:

$$a_k = \frac{a_{k-1} + b_{k-1}}{2}, \quad b_k = \frac{a_{k-1} - b_{k-1}}{2}$$

for $k \geq 1$. He repeated this process n times for some n . After the final step, the two numbers on the board were 20 and 25. If one of the original numbers was 720, find the other number.

Answer. The other number is 80.

Solution. By induction, we have that if $n = 2k$, then

$$(a_{2k}, b_{2k}) = \left(\frac{a_0 + b_0}{2^k}, \frac{a_0 - b_0}{2^k} \right).$$

Otherwise if $n = 2k + 1$, then

$$(a_{2k+1}, b_{2k+1}) = \left(\frac{a_0 + b_0}{2^{k+1}}, \frac{a_0 - b_0}{2^{k+1}} \right)$$

Now if n was even, then if $a_0 = 720$, then we have $\frac{720}{2^k} = 20$ or if $b_0 = 720$, then we have $\frac{720}{2^k} = 25$, both of which has no integer solutions k .

Now if n is odd, then if $a_0 = 720$, then $25 + 20 = \frac{720}{2^k}$. Similarly if $b_0 = 720$, then $25 - 20 = \frac{720}{2^k}$. Solving this yields only $k = 4$ for the former case.

Thus $\frac{720 + b_0}{2^{4+1}} = 25$ which means that $b_0 = 80$.

§6.6 Beginner Problem 6

Suppose for some positive integer n , its factorial $n!$ is written as a sum of some smaller factorials, such that no three of the factorials in the sum are equal. What is the maximum value of n ?

Note: A factorial of a positive integer n is defined to be $n! = 1 \times 2 \times \cdots \times n$.

Answer. The maximum value of n is 3.

Solution. We have $3! = 1! + 1! + 2! + 2!$ and $4! > 2(1! + 2! + 3!)$. Now, suppose that $k! > 2(1! + 2! + \cdots + (k-1)!)$ for some positive integer $k \geq 4$. We have

$$(k+1)! > 3k! > 2(1! + 2! + \cdots + k!)$$

hence by induction, we get that

$$n! > 2(1! + 2! + \cdots + (n-1)!)$$

for all positive integers $n \geq 4$.

§6.7 Pioneer Problem 1

Suppose $a_1, a_2, \dots, a_{10}$ are ten real numbers such that the numbers

$$(a_1 + 0.25)^2, (a_2 + 0.25)^2, \dots, (a_{10} + 0.25)^2$$

can be rearranged back to $a_1, a_2, \dots, a_{10}$.

Prove that $a_1 = a_2 = a_3 = \cdots = a_{10} = 0.25$.

Solution.

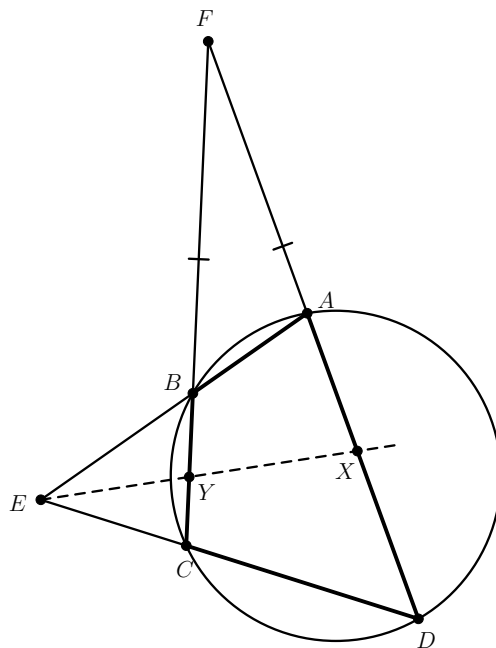
$$\begin{aligned} & (a_1 + 0.25)^2 + (a_2 + 0.25)^2 + \cdots + (a_{10} + 0.25)^2 = a_1 + a_2 + \cdots + a_{10} \\ \implies & (a_1 - 0.25)^2 + (a_2 - 0.25)^2 + \cdots + (a_{10} - 0.25)^2 = 0 \\ \implies & a_1 = a_2 = \cdots = a_{10} = 0.25 \end{aligned}$$

§6.8 Pioneer Problem 2

Suppose $ABCD$ is a cyclic quadrilateral such that the lines AB and CD intersect at E , and the lines AD and BC intersect at F . The internal angle bisector of angle $\angle AED$ intersects the lines AD and BC at X and Y respectively.

Prove that $FX = FY$.

Solution.



By using the exterior angles of $\triangle EXD$ and $\triangle EBY$, we get

$$\begin{aligned}\angle FXY &= \angle XED + \angle XDE = \angle XED + \angle ADC, \\ \angle FYX &= \angle BEY + \angle YBE = \angle AEX + \angle EBC.\end{aligned}$$

Given that $ABCD$ is a cyclic quadrilateral, $\angle EBC = \angle ADC$. Since EX is the angle bisector of $\angle AED$, $\angle AEX = \angle XED$. Therefore,

$$\angle FYX = \angle AEX + \angle EBC = \angle XED + \angle ADC = \angle FXY,$$

so triangle FXY is isosceles and $FX = FY$.

§6.9 Pioneer Problem 3

Find all pairs of positive integers (x, y) such that

$$1! + 2! + \cdots + x! = 1 + 2 + \cdots + y.$$

Note: $n!$ is defined to be the factorial $n! = 1 \times 2 \times \cdots \times n$.

Solution. We claim that only $(1, 1)$, $(2, 2)$ and $(5, 17)$ works.

For $x \leq 6$, we case bash to find corresponding y . Only $(1, 1)$, $(2, 2)$ and $(5, 17)$ works.

For $x \geq 7$, note that since $7 \mid x!$, so:

$$\text{LHS} \equiv 1! + 2! + 3! + 4! + 5! + 6! \equiv 5 \pmod{7}$$

But $\frac{n(n+1)}{2} \equiv 0, 1, 3, 6 \pmod{7}$. So no $x \geq 7$ works.

§6.10 Pioneer Problem 4

Let A, B be different positive real numbers. Ivan first writes A and B on the whiteboard. Then, in each operation, he can take two numbers x and y ($x \neq y$) on the whiteboard and write either $|x - y|$ or $x \div y$ on the whiteboard.

Can Ivan eventually write:

- i. $A \times B$?
- ii. $A + B$?

Answer. (i) Yes. (ii) Yes.

Solution. (i) We split into 2 cases.

Case 1: $A \times B \neq 1$. Ivan can do the following:

1. Write $A \div B$.
2. Write $(A \div B) \div A$. [If $(A \div B) = A$, then $B = 1$ and $A \times B = A$.]
3. Write $A \div ((A \div B) \div A) = A \times B$. [If $A = ((A \div B) \div A)$, then $A \times B = 1$, contradiction.]

Case 2: $A \times B = 1$. Notice that $A, B \neq 1$. WLOG, assume $A > 1$ and $B < 1$. Ivan can do the following:

1. Write $A \div B$.
2. Write $(A \div B) - A$. [$(A \div B) > A$]
3. Write $(A \div B) \div B$. [If $(A \div B) = B$, then $B = 1$, contradiction.]
4. Write $((A \div B) \div B) - A$. [$((A \div B) \div B) > A$]
5. Write $((A \div B) \div B) - A \div ((A \div B) - A)$. [$((A \div B) \div B) - A > ((A \div B) - A)$]
6. Write $((A \div B) \div B) - A \div ((A \div B) - A) - A = A \times B$.

(ii) We split into 3 cases.

Case 1: One of A and B is 1. WLOG, assume $A = 1$ and $B \neq 1$. Ivan can do the following:

1. Write $A \div B$.
2. Write $B \div (A \div B)$. [If $B = (A \div B)$, then $B = 1$, contradiction.]
3. Write $|(B \div (A \div B)) - A|$. [If $(B \div (A \div B)) = A$, then $B = 1$, contradiction.]
4. Write $|B - A|$.
5. Write $(|(B \div (A \div B)) - A|) \div (|B - A|) = A + B$. [$A + B > 1$]

Case 2: $A, B, A + B \neq 1$. WLOG, assume $A > B$. Ivan can do the following:

1. Write $B \div A$.

2. Write $(B \div A) \div B$. [If $(B \div A) = B$, then $A = 1$, contradiction.]
3. Write $A \div ((B \div A) \div B)$. [If $A = ((B \div A) \div B)$, then $A = 1$, contradiction.]
4. Write $A \div B$.
5. Write $(A \div B) \div A$. [If $(A \div B) = A$, then $B = 1$, contradiction.]
6. Write $B \div ((A \div B) \div A)$. [If $B = ((A \div B) \div A)$, then $B = 1$, contradiction.]
7. Write $(A \div ((B \div A) \div B)) - (B \div ((A \div B) \div A))$. $[(A \div ((B \div A) \div B)) > (B \div ((A \div B) \div A))]$
8. Write $A - B$.
9. Write $((A \div ((B \div A) \div B)) - (B \div ((A \div B) \div A))) \div (A - B) = A + B$. $[A + B \neq 1]$

Case 3: $A, B \neq 1$ and $A + B = 1$. Ivan can do the following:

1. Write $B \div A$.
2. Write $(B \div A) \div B = \frac{1}{A}$. [If $(B \div A) = B$, then $A = 1$, contradiction.]
3. Write $\left| \frac{1}{A} - (B \div A) \right|$. [If $\frac{1}{A} = (B \div A)$, then $B = 1$, contradiction.] Then

$$\begin{aligned}
 \left| \frac{1}{A} - \frac{B}{A} \right| &= \frac{1 - B}{A} \\
 &= \frac{A}{A} \\
 &= 1 \\
 &= A + B.
 \end{aligned}$$

§6.11 Pioneer Problem 5

Let n be an even positive integer. Janson has $3n$ coins with heads facing up, arranged into a rectangle with 3 rows and n columns. He wants to flip some coins such that the following is true: if any row is removed, exactly half of the columns will contain exactly one head and one tail.

Determine all n such that this is possible.

Solution 1. The answer is all n divisible by 4.

To show that these are possible, consider the following:

- First row is all heads
- Second row is $n/2$ heads followed by $n/2$ tails
- Third row is $n/4$ heads, $n/4$ tails, $n/4$ heads, $n/4$ tails

which clearly works.

Now we show that n must be divisible by 4. Observe that if all coins in a column are flipped, the property is still preserved. Therefore, we may assume the first row is all heads. Then, there must be $n/2$ tails in the second row and $n/2$ tails in the third row. Let the number of columns with 2 heads be a , then we obtain

$$\frac{n}{2} = 2 \cdot \frac{n}{2} - 2a$$

hence $\frac{n}{2}$ is even, so n is a multiple of 4.

Solution 2. Use the construction as before to show multiples of 4 work. Observe that there are exactly $3n/2$ pairs of coins in the same column, where the first coin is heads and the second is tails. However, in every column, the number of such pairs is either 0 or 2, so the total number of such pairs, $3n/2$, must be even.

§6.12 Pioneer Problem 6

Find all positive real numbers a, b, c such that:

- i. a, b, c are all greater than or equal to 3.
- ii. $ab + bc + ca + 9 = 3a + 4b + 5c$.

Answer. $(a, b, c) = (3, 3, 3)$ only.

Solution. Let $a = x + 3, b = y + 3, c = z + 3$ where $x, y, z \geq 0$. We have

$$\begin{aligned} ab + bc + ca + 9 &= 3a + 4b + 5c \\ (x+3)(y+3) + (y+3)(z+3) + (z+3)(x+3) + 9 &= 3(x+3) + 4(y+3) + 5(z+3) \\ xy + yz + zx + 3x + 2y + z &= 0 \end{aligned}$$

but $xy + yz + zx + 3x + 2y + z \geq 0$ with equality iff $x = y = z = 0$.

§6.13 Junior Problem 1

Find all arithmetic progressions of positive integers $a_1, a_2, \dots$ such that for all n ,

$$\gcd(a_n, a_{2n}) = n$$

Note: $\gcd(x, y)$ denotes the greatest common divisor of x and y .

Answer. The only sequence is the sequence $\{1, 2, 3, 4, \dots\}$ which clearly works as $\gcd(n, 2n) = n$.

Solution. Denote the arithmetic sequence as $\{a, a + d, a + 2d, \dots\}$. Now applying the problem condition, gives

$$\begin{aligned} n &= \gcd(a_n, a_{2n}) \\ &= \gcd(a + (n-1)d, a + (2n-1)d) \\ &= \gcd(nd, a + (2n-1)d) \\ &= \gcd(nd, a - d) \end{aligned}$$

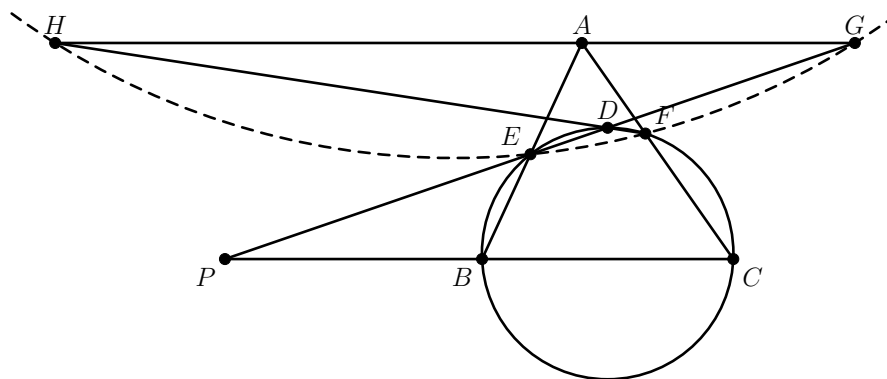
Since $a - d$ is fixed and is divisible by every positive integer n , this is enough to imply that $a - d = 0 \implies a = d$. Finally, $1 = \gcd(1, 2) = \gcd(a_1, a_2) = \gcd(a, a + d) = \gcd(a, 2a)$ implies that $a = d = 1$, which solves the problem.

§6.14 Junior Problem 2

Let ABC be an acute triangle with $AB < AC$. Let D be any point on the perpendicular bisector of BC . The circumcircle of BCD intersects AB at E and AC at F . The line DE and DF intersects the line parallel to BC through A at G and H .

Prove that $GHFE$ is cyclic quadrilateral.

Solution 1. Let ED intersect BC at P .



Then we have

$$\begin{aligned} \angle EFD &= \angle ECD = \angle BCD - \angle BCE = \angle CBD - \angle BCE \\ &= \angle DEC - \angle BCE = \angle EPB = \angle EGH, \end{aligned}$$

so $EFGH$ is cyclic.

Solution 2. Let DE, DF intersect BC at P, Q respectively, by Shooting Lemma we have

$$DP \cdot DE = DB^2 = DQ \cdot DF,$$

so $EFPQ$ is cyclic. By Reim's we have $EFGH$ is cyclic, since $PQ \parallel GH$.

§6.15 Junior Problem 3

There are 100 boxes. In each box, Ivan places a red ball into it, then for all $1 \leq i \leq 100$, he places i blue balls into the i -th box. Now, from each box, he picks one ball at random. What is the probability that he will get an odd number of red balls?

Solution. The answer is $\frac{1}{2}$. Suppose a ball has been picked from each box except for the first one, then when the ball is chosen from the first box, the two parities are equally likely to occur.

§6.16 Junior Problem 4

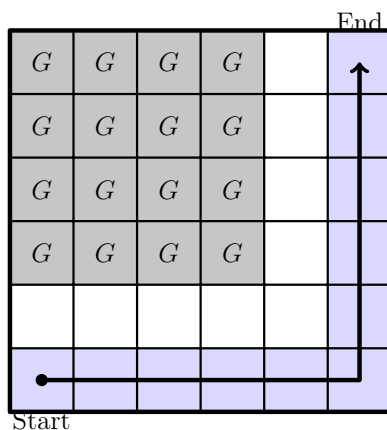
Given an n by n grid, some of the squares are placed with a garlic. A Dracula wants to move from the bottom left square to the top right square of the grid. In each step Dracula can only move to a square directly adjacent to it, and it cannot be directly adjacent or be on a square that contains garlic.

Given that Dracula succeeded in doing so, what is the maximum number of squares that can contain a garlic?

Solution. The answer is $(n - 2)^2$ for $n \geq 2$, and 0 for $n = 1$.

For $n = 1$, the only square is the one Dracula starts and ends on, so it cannot contain a garlic, and the answer is 0. From now on assume $n \geq 2$.

Let Dracula walk along the bottom row and then up the rightmost column, and place a garlic on every square outside the bottom two rows and the rightmost two columns. This gives $(n - 2)^2$ garlics, and none of them is on or adjacent to Dracula's path. The diagram below shows the case $n = 6$.



Label the squares (x, y) with $1 \leq x, y \leq n$, where $(1, 1)$ is the bottom left square, and for $d = 2, 3, \dots, 2n$ let D_d be the right-down diagonal of squares with $x + y = d$. There are $2n - 1$ such diagonals. Each step changes $x + y$ by exactly 1, so Dracula visits every diagonal.

The two diagonals D_2 and D_{2n} contain 1 square each, namely the start and the end, so they must be garlic free. We claim that each of the remaining $2n - 3$ diagonals D_d , $3 \leq d \leq 2n - 1$, contains at least 2 garlic free squares. Otherwise Dracula cannot land safely on the neighbouring diagonal:

- If $d \leq n + 1$, consider a square (x, y) of D_{d-1} that Dracula visits. Since $x + y = d - 1 \leq n$, both $(x + 1, y)$ and $(x, y + 1)$ are in the grid. They lie on D_d and are adjacent to Dracula, so both are garlic free.
- If $d \geq n + 1$, consider a square (x, y) of D_{d+1} that Dracula visits. Since $x + y = d + 1 \geq n + 2$, both $(x - 1, y)$ and $(x, y - 1)$ are in the grid. They lie on D_d and are adjacent to Dracula, so both are garlic free.

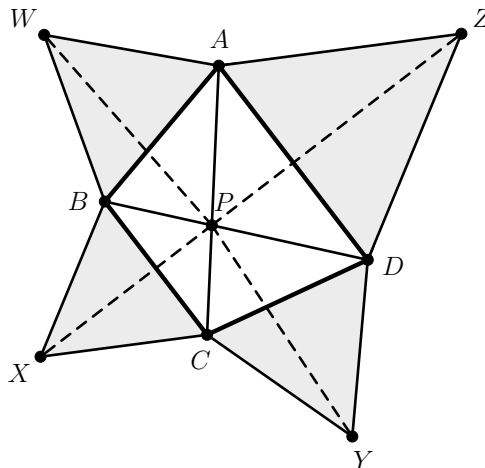
Hence at least $2 + 2(2n - 3) = 4n - 4$ squares are garlic free, and the number of garlicks is at most

$$n^2 - (4n - 4) = (n - 2)^2.$$

§6.17 Junior Problem 5

Let $ABCD$ be a cyclic quadrilateral with its diagonals AC and BD intersecting at P . Points W, X, Y, Z are constructed outside of quadrilateral $ABCD$, such that triangles WAB, XBC, YCD, ZDA are equilateral. Prove that $\angle WPZ + \angle YPX = 180^\circ$.

Solution.



Given that $ABCD$ is a cyclic quadrilateral, we know that $\triangle ABP$ is similar to $\triangle DCP$ and $\triangle ADP$ is similar to $\triangle BCP$. Since the triangles WAB, YCD, ZDA, XBC are equilateral,

$$\begin{aligned} \frac{WA}{AP} &= \frac{AB}{AP} = \frac{DC}{DP} = \frac{YD}{DP}, \\ \angle WAP &= 60^\circ + \angle BAP = 60^\circ + \angle CDP = \angle YDP, \\ \frac{ZA}{AP} &= \frac{AD}{AP} = \frac{BC}{BP} = \frac{XB}{BP}, \\ \angle ZAP &= 60^\circ + \angle DAP = 60^\circ + \angle CBP = \angle XBP, \end{aligned}$$

which implies that $\triangle WAP$ is similar to $\triangle YDP$ and $\triangle ZAP$ is similar to $\triangle XBP$. Hence $\angle WPA = \angle DPY$ and $\angle APZ = \angle BPX$, so

$$\angle WPZ = \angle WPA + \angle APZ = \angle DPY + \angle BPX.$$

Since B, P, D are collinear, the angles $\angle BPX, \angle XPY, \angle YPD$ form a straight line. Therefore,

$$\begin{aligned} \angle WPZ + \angle YPX &= \angle BPX + \angle XPY + \angle YPD \\ &= 180^\circ. \end{aligned}$$

§6.18 Junior Problem 6

Suppose that x is a real number greater than 1 such that $x^k \lfloor x^k \rfloor$ is an integer for $k = 1, 2, \dots, 2025$. Must x be an integer?

Note: $\lfloor x \rfloor$ denotes the floor function of x , the largest integer less than or equal to x .

Answer. No, x need not always be an integer.

Solution. We claim that $x = 2^{4049} + 0.5$ is a number such that $x^k \lfloor x^k \rfloor$ is an integer for $k = 1, 2, \dots, 2025$. By the binomial theorem,

$$(2^{4049} + 0.5)^k = \sum_{i=0}^k \binom{k}{i} (2^{4049})^i (0.5)^{k-i}$$

Since $k \leq 2025$, note that for $i \geq 1$, we have $(2^{4049})^i (0.5)^{k-i} = (2^{4049})^i (2)^{i-k} = 2^{4050i-k} \geq 2^{4050-2025} = 2^{2025}$.

Hence we may write $x^k = 2^{2025}n + 0.5^k$ and $\lfloor x^k \rfloor = 2^{2025}n$ for some positive integer n .

Now $x^k \lfloor x^k \rfloor = (2^{2025}n + 0.5^k)(2^{2025}n) = 2^{4050}n^2 + 2^{2025-k}n$ which is an integer as $2025 - k \geq 0$. This shows that x need not always be an integer.

§6.19 Senior Problem 1

An elephant is at number 0 and can make moves by going forward k steps, or going backwards 1 step. Determine the smallest m where there is a choice of k such that for all positive integers $a \leq 2025$, the elephant can reach number a in $\leq m$ moves.

Solution. The answer is 89, which is attained at $k = 45$ where it takes at most 45 moves to go to the smallest multiple of 45 which is at least a , then at most 44 backwards steps after that.

We show at least 89 is required for all k . Consider the largest m such that $1 + mk \leq 2025$. Then, at least $\lceil (1 + mk)/k \rceil = m + 1$ forward moves and at least $k - 1$ backwards moves are needed, so at least $m + k$ moves. Observe that

$$m + k = \left\lfloor \frac{2024}{k} \right\rfloor + k \geq \frac{2025}{k} + k - 1 \geq 2\sqrt{2025} - 1 = 89$$

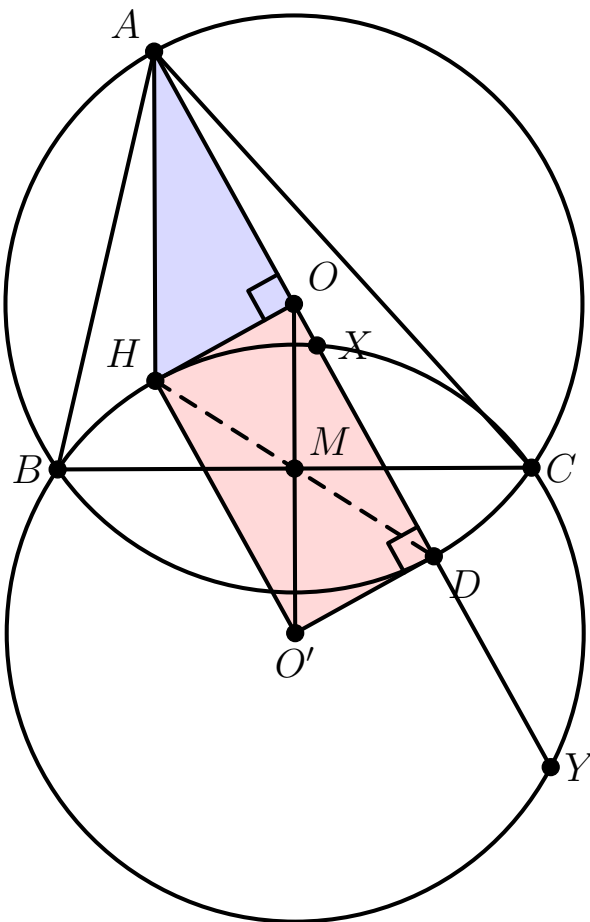
by AM-GM.

§6.20 Senior Problem 2

Let ABC be a triangle with circumcenter O and orthocenter H , and let AO intersect the circumcircle of BHC at two points X and Y . Prove that if $\angle AOH = 90^\circ$, then the midpoint of XY lies on the circumcircle of ABC .

Solution. Let O' be the center of (BHC) , D be the antipode of A in (ABC) , and M be the midpoint of BC . Since D lies on HM and AO , we have $\frac{AH}{OO'} = \frac{AH}{2OM} = \frac{AD}{2AO} = 1$, and since $AH \parallel OO'$ we have that $AHO'O$ is a parallelogram. Also, since $HO' \parallel AO \equiv OD$ and $HO' = AO = OD$, we have that $HODO'$ is a parallelogram.

Since $HO \perp OD$, we also have $O'D \perp OD$. Since the midpoint of the chord XY is the foot of perpendicular from O' to XY which is D and lies on (ABC) , we are done.



§6.21 Senior Problem 3

Let $n \geq 2$ be a positive integer. Determine in terms of n , the maximum value of

$$x_1(x_2 - x_1) + x_2(x_3 - x_2) + \cdots + x_{n-1}(x_n - x_{n-1})$$

over all positive reals $x_1, x_2, \dots, x_n$ satisfying $x_1 \leq x_2 \leq \cdots \leq x_n \leq n$.

Solution 1. The answer is $\frac{n(n-1)}{2}$, attained at $1, 2, \dots, n$.

Let $d_1 = x_1$, $d_2 = x_2 - x_1$, $d_3 = x_3 - x_2$, $\dots$, then the expression is equal to

$$\begin{aligned} & d_1 d_2 + (d_1 + d_2) d_3 + (d_1 + d_2 + d_3) d_4 + \cdots + (d_1 + d_2 + \cdots + d_{n-1}) d_n \\ &= \sum_{i < j} d_i d_j = \frac{1}{2} ((d_1 + d_2 + \cdots + d_n)^2 - (d_1^2 + d_2^2 + \cdots + d_n^2)) \\ &\leq \frac{1}{2} \left(n^2 - \frac{1}{n} \cdot n^2 \right) \end{aligned}$$

by QM-AM, proving the inequality.

Solution 2. Rewrite the expression as

$$-\frac{1}{2} \left((x_1 - 1)^2 + (x_2 - x_1 - 1)^2 + \cdots + (x_n - x_{n-1} - 1)^2 - x_n^2 + 2x_n - n \right) \leq \frac{x_n^2 - 2x_n + n}{2}$$

which is clearly $\leq \frac{n(n-1)}{2}$.

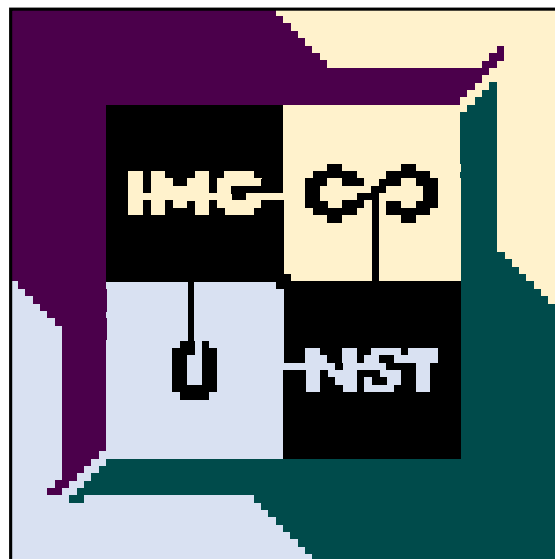
§6.22 Senior Problem 4

A tile is a collection of orthogonally connected unit squares. Does there exist a set S of 5 pairwise distinct tiles such that:

- All tiles in S are made up by the same number of squares;
- None of the tiles in S are rectangles;
- One can tile an $m \times n$ board using only tiles in S (rotation allowed, reflection not allowed, not using certain tiles is allowed) if and only if 2025 divides both m, n ?

Answer. Yes.

Solution. We consider the following 5 tiles:



The border is a 2025×2025 square.

I claim that all tiles must be part of a 2025×2025 subsquare and the tiles are arranged within their subsquare as shown above (including rotated variants).

We see that by looking at the one-square wide cavities of the beige tile, a beige piece immediately forces the purple and black tiles to fit in the position above, which in turn

forces the blue and dark green tiles to fit into a 2025×2025 subsquare. Similarly, a black tile or a blue tile also does the same thing. Hence, all beige, black and blue tiles must be part of a 2025×2025 subsquare.

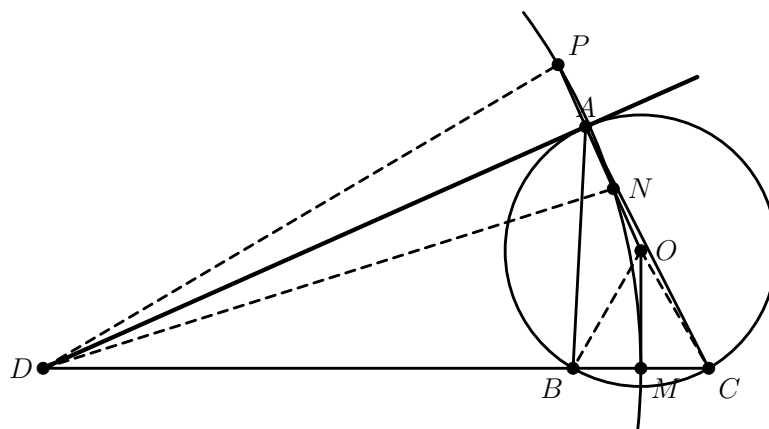
Next, purple forces beige and green can't fit with green and 2025×2025 squares, hence these 5 tiles work.

§6.23 Senior Problem 5

Let ABC be an acute triangle with circumcenter O . Let M, N be the midpoints of BC and AO , and the tangent to the circumcircle of ABC at A intersects line BC at D . Suppose $DM = DN$, find the value of $\angle A$.

Solution. The answer is 30° .

Let P be the reflection of N over A . Since $DA \perp OA$, we must have $DM = DN = DP$.



Since OM is tangent to (PNM) , by power of a point,

$$OM^2 = ON \cdot OP = \frac{3}{4}OA^2$$

$$OM = OB \cdot \frac{\sqrt{3}}{2} = OB \cos 30^\circ$$

Hence, $\angle A = \angle BOM = 30^\circ$.

§6.24 Senior Problem 6

Let S be a set of 2025 distinct integers. A subset $T \subseteq S$ is called progression-free if it contains no three elements in arithmetic progression.

Prove that S always contains a progression-free subset of size at least:

- i. 37.
- ii. 119.

Solution to (i). Consider the largest such set T . Then, every element $x \in S \setminus T$ cannot be added into T , so there exist distinct $a, b \in T$ such that a, b, x form an arithmetic progression in some order, i.e. $x \in \left\{2b - a, 2a - b, \frac{a+b}{2}\right\}$. Since there are only $\binom{|T|}{2}$ pairs of a, b , we obtain

$$2025 - |T| \leq 3 \binom{|T|}{2}.$$

If $|T| \leq 36$, then $|T| + 3 \binom{|T|}{2} \leq 1926$, contradiction. $\square$

Solution to (ii). Given n , let $f(n)$ denote the maximum integer such that every set of size n has a progression-free subset of size $\geq f(n)$.

Let $\alpha = \log_3(2)$. We will show by strong induction that $f(n) \geq n^\alpha$ for all n . For $n = 0, 1$ this obviously holds.

If all elements in S are the same mod 3, we can assume it is 0, then we repeatedly divide every element by 3 until otherwise. Now suppose S has a, b, c elements which are $0, 1, 2 \pmod{3}$ respectively, where $a + b + c = n$ and $a, b, c < n$. Since every arithmetic progression of length 3 must be either constant mod 3 or cover all residues mod 3, we may pick any two residues mod 3 and select the largest progression-free subset of those two sets. This means there is a progression-free subset of S with size at least

$$\max\{f(a) + f(b), f(b) + f(c), f(c) + f(a)\}.$$

Without loss of generality $a \geq b \geq c$, so $b \geq 1$ and this is at least $a^\alpha + b^\alpha$. Let $g(t) = t^\alpha + 1 - (t+2)^\alpha$. Then $g(1) = 2 - 3^\alpha = 0$ and

$$g'(t) = \alpha(t^{\alpha-1} - (t+2)^{\alpha-1}) > 0$$

as $\alpha < 1$, so $g(t) \geq 0$ for $t \geq 1$. Taking $t = \frac{a}{b}$,

$$\begin{aligned} a^\alpha + b^\alpha &= b^\alpha(t^\alpha + 1) \\ &\geq b^\alpha(t+2)^\alpha \\ &= (a+2b)^\alpha \\ &\geq (a+b+c)^\alpha \\ &= n^\alpha. \end{aligned}$$

This completes the induction.

Now, observe that

$$f(2025) \geq 2025 \cdot 2025^{\alpha-1} \geq 2025 \cdot (3^7)^{\alpha-1} = 2025 \cdot \frac{128}{2187} = 118.5 \dots$$

therefore $f(2025) \geq 119$. $\square$