

IMO NATIONAL SELECTION TEST 2

SAMPLE PAPER

SENIOR CATEGORY

Six questions · 7 marks per question · 42 marks

INSTRUCTIONS

1. This paper contains six essay questions. Answer all questions. Each question is worth 7 marks.
 2. Suggested duration: 4 hours.
 3. Show your working clearly. Marks may be awarded for correct reasoning even if the final answer is incorrect.
 4. The English questions are presented first, followed by the Bahasa Melayu questions. The solutions and suggested mark allocations follow both language sections.
 5. No calculator is required.
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ENGLISH

Each question is worth 7 marks.

Question 1

For positive real numbers a, b, c , prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Question 2

Let $a_1, a_2, \dots, a_n$ be any integers in a fixed order. Prove that some nonempty consecutive block of these integers has a sum divisible by n .

Question 3

Prove that 30 divides $n^5 - n$ for every integer n .

Question 4

A complete graph has every pair of its vertices joined by an edge. Each edge of a complete graph is coloured red or blue. Determine the smallest integer m such that every red-blue colouring of the edges of a complete graph on m vertices contains a triangle whose three edges have the same colour.

Question 5

Let ABC be an acute triangle with orthocenter H . Let H' be the reflection of H across line BC . Prove that A, B, C, H' lie on one circle.

Question 6

Determine all positive integer pairs (x, y) satisfying

$$x^2 + y^2 + 1 = 3xy.$$

BAHASA MELAYU

Setiap soalan bernilai 7 markah.

Soalan 1

Bagi nombor nyata positif a, b, c , buktikan bahawa

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Soalan 2

Andaikan $a_1, a_2, \dots, a_n$ ialah sebarang integer dalam suatu susunan tetap. Buktikan bahawa terdapat satu blok tidak kosong yang terdiri daripada sebutan-sebutan berturutan dan jumlahnya boleh dibahagi dengan n .

Soalan 3

Buktikan bahawa 30 membahagi $n^5 - n$ bagi setiap integer n .

Soalan 4

Dalam sebuah graf lengkap, setiap pasangan bucu disambungkan oleh satu sisi. Setiap sisi graf lengkap diwarnakan merah atau biru. Cari integer terkecil m supaya setiap pewarnaan merah-biru bagi sisi graf lengkap dengan m bucu mengandungi segi tiga yang ketiga-tiga sisinya mempunyai warna yang sama.

Soalan 5

Andaikan ABC segi tiga tirus dengan ortopusat H . Biarkan H' ialah pantulan H terhadap garis BC . Buktikan bahawa titik A, B, C, H' terletak pada satu bulatan.

Soalan 6

Cari semua pasangan integer positif (x, y) yang memenuhi

$$x^2 + y^2 + 1 = 3xy.$$

SOLUTIONS AND SUGGESTED MARK ALLOCATIONS

Each question is worth 7 marks. Accept any mathematically correct solution with clear reasoning.

Question 1

By Cauchy–Schwarz in Engel form,

$$\sum_{\text{cyc}} \frac{a}{b+c} = \sum_{\text{cyc}} \frac{a^2}{a(b+c)} \geq \frac{(a+b+c)^2}{a(b+c) + b(c+a) + c(a+b)} = \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Since $(a+b+c)^2 \geq 3(ab+bc+ca)$, the last expression is at least $3/2$. Therefore the inequality holds, with equality at $a = b = c$. **Suggested mark allocation:**

- 3 marks for the Cauchy–Schwarz step.
- 2 marks for using $(a+b+c)^2 \geq 3(ab+bc+ca)$.
- 2 marks for concluding the bound and identifying equality.

Question 2

Let $S_k = a_1 + \dots + a_k$ for $1 \leq k \leq n$. If any S_k is divisible by n , the first k terms give the required block. Otherwise, each of $S_1, \dots, S_n$ has a nonzero remainder modulo n . There are only $n-1$ possible nonzero remainders, so two partial sums, say S_i and S_j with $i < j$, have the same remainder. Then $S_j - S_i = a_{i+1} + \dots + a_j$ is divisible by n . This is a nonempty consecutive block. **Suggested mark allocation:**

- 2 marks for defining the partial sums.
- 2 marks for handling a partial sum divisible by n .
- 2 marks for the pigeonhole argument on nonzero remainders.
- 1 mark for identifying the resulting nonempty block.

Question 3

Modulo 2, $n^5 - n$ is even. Modulo 3, checking $n \equiv 0, 1, 2$ gives $n^5 \equiv n$. Modulo 5, if $5 \mid n$ the result is immediate; otherwise Fermat's little theorem gives $n^4 \equiv 1 \pmod{5}$, hence $n^5 \equiv n \pmod{5}$. Therefore $n^5 - n$ is divisible by 2, 3, and 5. These are pairwise coprime, so $30 \mid n^5 - n$. **Suggested mark allocation:**

- 1 mark for divisibility by 2.
- 2 marks for divisibility by 3.
- 2 marks for divisibility by 5.
- 2 marks for combining the pairwise coprime factors.

Question 4

For the upper bound, consider any vertex V in a graph on 6 vertices. Among its 5 incident edges, at least 3 have the same colour. Suppose these join V to A, B, C . If any of AB, BC, CA has that same colour, it forms a monochromatic triangle with V . Otherwise all three are the other colour, and ABC is a monochromatic triangle. Thus $m \leq 6$. For 5 vertices, colour the edges of a pentagon red and its diagonals blue. Neither colour contains a triangle, so m cannot be 5. Therefore the smallest value is $\boxed{6}$. **Suggested mark allocation:**

- 2 marks for finding three incident edges of one colour.
- 2 marks for the two cases among their other endpoints.
- 1 mark for concluding that 6 vertices always suffice.
- 2 marks for the 5-vertex counterexample.

Question 5

Since $BH \perp AC$ and $CH \perp AB$, $\angle HBC = 90^\circ - C$ and $\angle HCB = 90^\circ - B$. Thus $\angle BHC = 180^\circ - (90^\circ - C) - (90^\circ - B) = 180^\circ - A$. Reflection across BC preserves angles and fixes B, C , so $\angle BH'C = 180^\circ - A$. Since $\angle BAC = A$, the opposite angles of $ABCH'$ sum to 180° . Hence A, B, C, H' are concyclic. **Suggested mark allocation:**

- 2 marks for finding $\angle BHC = 180^\circ - A$.
- 2 marks for transferring the angle under reflection.
- 2 marks for using supplementary opposite angles.
- 1 mark for the cyclic conclusion.

Question 6

Assume $x \geq y$. If $x = y$, the equation gives $x = y = 1$. If $y = 1$, then $x^2 - 3x + 2 = 0$, so $x = 1$ or 2 . For $x > y > 1$, regard the equation as a quadratic in x : $X^2 - 3yX + (y^2 + 1) = 0$. Its other root is $x' = 3y - x = (y^2 + 1)/x$, a positive integer. Since $x \geq y + 1$, we have $xy \geq y(y + 1) > y^2 + 1$, so $x' < y$. The pair (y, x') is another positive solution with smaller sum. Repeating this descent reaches $(1, 1)$ or $(2, 1)$. Conversely, from any solution (u_k, u_{k+1}) , the other integer root is $3u_{k+1} - u_k$, so (u_{k+1}, u_{k+2}) is also a solution. Reversing the descent therefore gives all solutions as (u_k, u_{k+1}) or their reversals, where $u_1 = u_2 = 1$ and $u_{k+2} = 3u_{k+1} - u_k$ for $k \geq 1$. **Suggested mark allocation:**

- 1 mark for the equal-coordinate/base cases.
- 2 marks for obtaining the other integer root x' .
- 2 marks for proving $0 < x' < y$ and that descent preserves the equation.
- 2 marks for reversing the descent and stating the full sequence of solutions.